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THE MATHEMATICAL ASSOCIATION

AN ASSOCIATION OF TEACHERS AND STUDENTS OF ELEMENTARY MATHEMATICS

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'I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.'
BACON

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THE TEACHING OF STATISTICS IN SCHOOLS

BY FREDA CONWAY

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All schools will not find it necessary to cater for each of these needs, but those that wish to make such provision will probably want a single comprehensive course, if one can be devised. Since statistics involves the application of mathematics to scientific method, as well as to the subject matter of individual sciences, the first requirement for a comprehensive course is that it should satisfy some of the common needs of all scientists. There can be little doubt that this requirement is best met by a statistical method course in which statistics is developed as a set of techniques for measuring and analysing the variation between members of the same group.

This variation is a most important characteristic of all natural and social groups. Men vary in height, weight and ability: they work and live in groups of different sizes: plants and animals of the same species are never exactly alike. Less obvious but equally important is the fact that this variation conforms to a pattern. In the first place statistical methods are necessary to describe these patterns of variation. A statistics course might begin with frequency distributions, averages and measures of dispersion. From these it is natural to extend the discussion to bivariate distributions and include the topics of correlation and regression analysis. Also since most natural and social groups are large, sampling must be used to obtain information concerning them.

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All schools will not find it necessary to cater for each of these needs, but those that wish to make such provision will probably want a single comprehensive course, if one can be devised. Since statistics involves the application of mathematics to scientific method, as well as to the subject matter of individual sciences, the first requirement for a comprehensive course is that it should satisfy some of the common needs of all scientists. There can be little doubt that this requirement is best met by a statistical method course in which statistics is developed as a set of techniques for measuring and analysing the variation between members of the same group.

This variation is a most important characteristic of all natural and social groups. Men vary in height, weight and ability: they work and live in groups of different sizes: plants and animals of the same species are never exactly alike. Less obvious but equally important is the fact that this variation conforms to a pattern. In the first place statistical methods are necessary to describe these patterns of variation. A statistics course might begin with frequency distributions, averages and measures of dispersion. From these it is natural to extend the discussion to bivariate distributions and include the topics of correlation and regression analysis. Also since most natural and social groups are large, sampling must be used to obtain information concerning them.

The mathematical content of a course of this kind can be varied to suit the needs of the students taking it. Very little mathematics is really essential: the techniques employed could be limited to arithmetic and a little simple algebra and geometry. It can be made suitable for those biologists and social scientists who describe themselves as " not very good at mathematics ". On the other hand if the course is to be intelligible to these non-mathematicians it is essential that it should be illustrated with real data: very little is to be gained by calculating correlation coefficients for values of X and Y. Experiments with dice, pennies and random numbers could also be used to illustrate the concept of a frequency distribution.

Though this course would be designed to meet the common needs of scientists, it would provide an excellent introduction to statistics for mathematicians at school level.

The present day tendency among teachers of mathematics appears to be to teach all topics as science before teaching them as mathematics. Stage A Geometry is " numerical, experimental and practical ". " The kind of understanding that arises from experience should precede the development of mathematical techniques ". " Mechanism must come before mechanics. The mathematics of a subject is a super-structure to be built upon a foundation of clear ideas." An introductory course of statistical methods such as that described above would be in accord with the best traditions of mathematical teaching.

A course of this kind would have another important advantage. Providing the data used to illustrate the statistical methods involved only fairly simple measurements such as the heights and weights of schoolboys and the sizes of towns and classes, it would require little knowledge of any particular science by either pupil or teacher. Nevertheless if courses of this type are to be given by teachers of mathematics it may be necessary for a greater proportion of these specialists to study one of the biological or social sciences at least to intermediate or subsidiary level.

The statistical methods course sketched here would not be suitable for all pupils, nor of course does it satisfy all the needs described above. The advocates of school statistics have always stressed its importance as a preparation for citizenship and G.C.E. syllabuses often include such topics as price indices, mortality rates and estimates of the seasonal variation in economic time series. These topics of social statistics differ from the more general methods described above. They involve the application of statistics to the subject matter of the social sciences, and consequently their study involves some knowledge of current social theories and existing social conditions. Price indices and standardised mortality rates are both examples of weighted averages, but their usefulness cannot be judged solely in mathematical terms.

Formal courses in social statistics may well be unsuitable for schools: certainly they are more suitable for older than for younger students. Perhaps the most that any school should attempt is to include some of the elements of social statistics in the mathematics or social studies lessons: but this will only be possible if either some teachers of mathematics study social statistics as part of their preparation for teaching, or if teachers of social studies are also competent teachers of elementary mathematics.

University of Leicester F.C.

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CARTESIAN GEOMETRY OF THE TRIANGLE AND HEXAGON

By R. Sibson

1. *Introduction.* The present paper is a sequel to two others which have appeared in the *Gazette* (July, 1941 and December, 1957) and in which the possibility of attacking a number of familiar triangle theorems by cartesian methods was successfully investigated. The same method is here applied to Morley's and Pascal's Theorems.

2. We begin by establishing two lemmas.

A. The chord joining the points $(R\cos\alpha, R\sin\alpha)$ and $(R\cos\beta, R\sin\beta)$ on the circle $x^2+y^2=R^2$ has as its equation

$$x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=R\cos\frac{\alpha-\beta}{2}$$

For two other points the angular coordinates of which are γ and δ the corresponding equation is

$$x\cos\frac{\gamma+\delta}{2}+y\sin\frac{\gamma+\delta}{2}=R\cos\frac{\gamma-\delta}{2}$$

Solving these for the coordinates of their point of intersection, we obtain

$$x\sin\frac{\alpha+\beta-\gamma-\delta}{2}=R\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\gamma-\delta}{2}-\sin\frac{\gamma+\delta}{2}\cos\frac{\alpha-\beta}{2}\right),$$

and if we denote $\frac{\alpha+\beta+\gamma+\delta}{2}$ by σ, this can easily be reduced to

$$x\sin\frac{\alpha+\beta-\gamma-\delta}{2}=\frac{R}{2}\{-\sin(\sigma-\alpha)-\sin(\sigma-\beta)+\sin(\sigma-\gamma)+\sin(\sigma-\delta)\}.$$

Similarly

$$y\sin\frac{\alpha+\beta-\gamma-\delta}{2}=\frac{R}{2}\{\cos(\sigma-\alpha)+\cos(\sigma-\beta)-\cos(\sigma-\gamma)-\cos(\sigma-\delta)\}.$$

B. (a) If in an Argand diagram three points start from a common origin and are given displacements $z(=x+iy)$, $z\operatorname{cis}\frac{2\pi}{3}$, $z\operatorname{cis}\frac{4\pi}{3}$, their final positions will obviously be at the vertices of an equilateral triangle. If they are subsequently given further displacements z', $z'\operatorname{cis}\frac{2\pi}{3}$, $z'\operatorname{cis}\frac{4\pi}{3}$ respectively, their new positions will still be at the vertices of an equilateral triangle, and so on for any number of similar sets of displacements.

(b) The same is true of a set of displacements 0, z, $z\operatorname{cis}\frac{\pi}{3}$ in the same cyclic sense, for this can be viewed as a set of displacements of type (a) coupled with a parallel translation of common magnitude applied to all three points.

It follows that any combination of sets of displacements of types (a) and (b) will still leave the points at the vertices of an equilateral triangle.

3. We now revert to the notation of the previous papers, in which the circle $x^2+y^2=R^2$ is the circumcircle of a triangle ABC of which the vertices A, B, and C have angular coordinates θ_1, θ_2, and θ_3 respectively, and $0<\theta_1<\theta_2<\theta_3<2\pi$. The internal trisectors of the angle C meet the circumference of the circle again in points of which the angular coordinates are $\frac{2\theta_1+\theta_2}{3}$ and $\frac{\theta_1+2\theta_2}{3}$, the former being nearer to A. The corresponding points between B and C have angular coordinates $\frac{2\theta_2+\theta_3}{3}$ and $\frac{\theta_2+2\theta_3}{3}$, and those between C and A,

for which we are obliged to define A by the angle $2\pi + \theta_1$, $\dfrac{2\theta_3 + \theta_1 + 2\pi}{3}$, $\dfrac{\theta_3 + 2\theta_1 + 4\pi}{3}$.

Let M_1, M_2, and M_3 be the points of intersection of the pairs of trisectors nearer to each side of the triangle. Then using lemma A we obtain the coordinates of M_1 by writing

$$\alpha = \theta_2, \quad \beta = \frac{2\theta_3 + \theta_1 + 2\pi}{3}, \quad \gamma = \theta_3, \quad \delta = \frac{\theta_1 + 2\theta_2}{3} \qquad \text{whence}$$

$$\sigma = \frac{2\theta_1 + 5\theta_2 + 5\theta_3 + 2\pi}{6}.$$

For M_1 this gives

$$x \sin\frac{\theta_2 + 2\pi - \theta_3}{6} = \frac{R}{2}\left(-\sin\frac{2\theta_1 - \theta_2 + 5\theta_3 + 2\pi}{6} - \sin\frac{5\theta_2 + \theta_3 - 2\pi}{6} + \sin\frac{2\theta_1 + 5\theta_2 - \theta_3 + 2\pi}{6} + \sin\frac{\theta_2 + 5\theta_3 + 2\pi}{6}\right)$$

$$y \sin\frac{\theta_2 + 2\pi - \theta_3}{6} = \frac{R}{2}\left(\cos\frac{2\theta_1 - \theta_2 + 5\theta_3 + 2\pi}{6} + \cos\frac{5\theta_2 + \theta_3 - 2\pi}{6} - \cos\frac{2\theta_1 + 5\theta_2 - \theta_3 + 2\pi}{6} - \cos\frac{\theta_2 + 5\theta_3 + 2\pi}{6}\right),$$

from which

$$x = R\left(-\cos\frac{\theta_1 + \theta_2 + \theta_3 + \pi}{3} + \cos\frac{\theta_1 + 2\theta_2}{3} - \cos\frac{\theta_1 + 2\theta_3 - \pi}{3} + \cos\frac{2\theta_2 + \theta_3 + \pi}{3} + \cos\frac{\theta_2 + 2\theta_3 - \pi}{3}\right)$$

with a corresponding expression in sines for y. Hence

$$\frac{\overrightarrow{OM_1}}{R} = \frac{x + iy}{R} = -\operatorname{cis}\frac{\theta_1 + \theta_2 + \theta_3 + \pi}{3} + \operatorname{cis}\frac{\theta_1 + 2\theta_2}{3} - \operatorname{cis}\frac{\theta_1 + 2\theta_3 - \pi}{3} + \operatorname{cis}\frac{2\theta_2 + \theta_3 + \pi}{3} + \operatorname{cis}\frac{\theta_2 + 2\theta_3 - \pi}{3}.$$

The corresponding expressions for $\overrightarrow{OM_2}$ and $\overrightarrow{OM_3}$ are obtainable by applying the "cyclic" exchange $\theta_1 \to \theta_2 \to \theta_3 \to \theta_1 + 2\pi$ in the above and we can tabulate the results as under.

$$\frac{\overrightarrow{OM_1}}{R} = -\operatorname{cis}\frac{\theta_1 + \theta_2 + \theta_3 + \pi}{3} + \operatorname{cis}\frac{\theta_1 + 2\theta_2}{3} + \operatorname{cis}\frac{\theta_1 + 2\theta_3 + 2\pi}{3} + \operatorname{cis}\frac{\theta_2 + 2\theta_3 - \pi}{3} + 0 + 0 + \operatorname{cis}\frac{\theta_3 + 2\theta_2 + \pi}{3};$$

$$\frac{\overrightarrow{OM_2}}{R} = -\operatorname{cis}\frac{\theta_1+\theta_2+\theta_3+3\pi}{3} + 0 + \operatorname{cis}\frac{\theta_1+2\theta_3+3\pi}{3} + \operatorname{cis}\frac{\theta_2+2\theta_3}{3}$$
$$+ \operatorname{cis}\frac{\theta_2+2\theta_1}{3} + \operatorname{cis}\frac{\theta_3+2\theta_1+3\pi}{3} + 0 ;$$

$$\frac{\overrightarrow{OM_3}}{R} = -\operatorname{cis}\frac{\theta_1+\theta_2+\theta_3+5\pi}{3} + \operatorname{cis}\frac{\theta_1+2\theta_2+5\pi}{3} + 0 + 0 + \operatorname{cis}\frac{\theta_2+2\theta_1+\pi}{3}$$
$$+ \operatorname{cis}\frac{\theta_3+2\theta_1+4\pi}{3} + \operatorname{cis}\frac{\theta_3+2\theta_2}{3} .$$

Scrutiny of the corresponding terms in these three expressions reveals that they are all sets of types (a) or (b) of lemma **B**, and therefore $M_1M_2M_3$ is an equilateral triangle. Thus Morley's Theorem is proved, and by reverting to the expressions for the coordinates of M_1, M_2 and M_3 we can calculate the length of the sides of the Morley triangle and their directions.

4. Giving M_1, M_2, and M_3 coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) we have, re-arranging terms for convenience,

$$\frac{x_1-x_2}{3} = \cos\frac{\theta_1+\theta_2+\theta_3+3\pi}{3} - \cos\frac{\theta_1+\theta_2+\theta_3+\pi}{3} + \cos\frac{\theta_1+2\theta_2}{3}$$
$$+ \cos\frac{\theta_1+2\theta_3+2\pi}{3} + \cos\frac{\theta_1+2\theta_3}{3} + \cos\frac{2\theta_2+\theta_3+\pi}{3}$$
$$- \cos\frac{\theta_3+2\theta_1+3\pi}{3} - \cos\frac{\theta_2+2\theta_3+2\pi}{3} - \cos\frac{\theta_2+2\theta_3}{3}$$
$$+ \cos\frac{\theta_2+2\theta_1+3\pi}{3} .$$

Combining terms 1 and 2, 4 and 5, 6 and 7, 8 and 9, we obtain an expression which can be reduced easily to

$$x_1 - x_2 = 8R\sin\frac{\theta_1+\theta_2+\theta_3+2\pi}{3}\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3} .$$

A similar reduction yields

$$y_1 - y_2 = -8R\cos\frac{\theta_1+\theta_2+\theta_3+2\pi}{3}\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3} .$$

Hence $M_1M_2 = 8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3} = M_2M_3 = M_3M_1$ similarly.

Also M_1M_2 is perpendicular to the radius of the circle to the point $\frac{\theta_1+\theta_2+\theta_3+2\pi}{3}$, and thus the orientation of the triangle $M_1M_2M_3$ is determined.

5. In the *Gazette* for February, 1938, W. J. Dobbs gives an exhaustive description of the complete Morley figure obtained by consider-

ing not only the " internal " trisectors of the angles of the triangle but also the " external " trisectors obtained by trisecting, e.g. $2\pi + A$ and $4\pi + A$. (loc. cit., p. 50, Fig. 1, which, however, needs slight amendment to make all the angles at O add up to 2π.)

The method pursued in the present investigation provides an alternative means of obtaining the properties of the figure. For example the trisectors of the angle $2\pi + A$ meet the circumference of the circle again in the points whose angular coordinates are

$$\frac{2\theta_2 + \theta_3 + 2\pi}{3} \quad \text{and} \quad \frac{\theta_2 + 2\theta_3 + 4\pi}{3}.$$

Thus for the point M_1' where the corresponding trisectors of $2\pi + B$ and $2\pi + C$ meet to give a point analogous to M_1 we have, reverting to the notation of Lemma **A**.

$$\begin{aligned} \alpha &= \theta_2 \\ \beta &= \frac{2\theta_3 + \theta_1 + 4\pi}{3} \\ \gamma &= \theta_3 \\ \delta &= \frac{\theta_1 + 2\theta_2 + 4\pi}{3}. \end{aligned}$$

These may be replaced by

$$\begin{aligned} \alpha &= \theta_2 + 2\pi &&= \theta_2' \\ \beta &= \frac{2(\theta_3 + 4\pi) + \theta_1 + 2\pi}{3} &&= \frac{2\theta_3' + \theta_1' + 2\pi}{3} \\ \gamma &= \theta_3 + 4\pi &&= \theta_3' \\ \delta &= \frac{\theta_1 + 2(\theta_2 + 2\pi)}{3} &&= \frac{\theta_1' + 2\theta_2'}{3}. \end{aligned}$$

These values are formally similar to those in para. 3 from which the coordinates of M_1 were derived. We can therefore write at once

$$\frac{\overrightarrow{OM_1'}}{R} = -\operatorname{cis}\frac{(\theta_1' + \theta_2' + \theta_3' + \pi)}{3} + \ldots$$

where $\theta_1' = \theta_1$, $\theta_2' = \theta_2 + 2\pi$, $\theta_3' = \theta_3 + 4\pi$.

Hence

$$\frac{\overrightarrow{OM_1'}}{R} = -\operatorname{cis}\frac{\theta_1 + \theta_2 + \theta_3 + \pi}{3} + \operatorname{cis}\frac{\theta_1 + 2\theta_2 + 4\pi}{3} + \operatorname{cis}\frac{\theta_1 + 2\theta_3 + 4\pi}{3}$$
$$+ \operatorname{cis}\frac{2\theta_2 + \theta_3 + 3\pi}{3} + \operatorname{cis}\frac{\theta_2 + 2\theta_3 + 3\pi}{3}$$

and the same considerations as before at once establish that triangle $M_1'M_2'M_3'$ is equilateral and that its sides are parallel to those of triangle $M_1M_2M_3$.

6. As a bridge from the triangle to the hexagon, one property of the cyclic quadrilateral may be considered.

Let $ABCD$ be a cyclic quadrilateral, inscribed in the circle $x^2+y^2=R^2$, with A, B, C, D defined by angular coordinates θ_1, θ_2, θ_3, θ_4 in ascending order of magnitude.

Then $AB=2R\sin\dfrac{\theta_2-\theta_1}{2}$, etc. Hence

$$\begin{aligned}
&AB\,.\,CD+AD\,.\,BC\\
&=4R^2\left[\sin\frac{\theta_2-\theta_1}{2}\sin\frac{\theta_4-\theta_3}{2}+\sin\frac{\theta_4-\theta_1}{2}\sin\frac{\theta_3-\theta_2}{2}\right]\\
&=2R^2\left[\cos\frac{\theta_4-\theta_3-\theta_2+\theta_1}{2}-\cos\frac{\theta_4-\theta_3+\theta_2-\theta_1}{2}\right.\\
&\qquad\left.+\cos\frac{\theta_4-\theta_3+\theta_2-\theta_1}{2}-\cos\frac{\theta_4+\theta_3-\theta_2-\theta_1}{2}\right]\\
&=4R^2\sin\frac{\theta_4-\theta_2}{2}\sin\frac{\theta_3-\theta_1}{2}=AC\,.\,BD
\end{aligned}$$

and this is Ptolemy's Theorem.

7. To prove Pascal's Theorem, for the limited case of a hexagon inscribed in a circle, whence generalisation by projection is possible, we define two triads of points on the circle $x^2+y^2=R^2$ by the angular coordinates α, β, γ; θ, ϕ, ψ. It will be convenient to refer to the points by these letters also. The hexagon considered in the first place is that obtained by taking the points in the order $\alpha\phi\gamma\theta\beta\psi$ so that its opposite pairs of sides are $\beta\psi, \gamma\phi$; $\gamma\theta, \alpha\psi$; $\alpha\phi, \beta\theta$. Denote the points of intersection of these by (ξ_i, η_i), $i=1,2,3$, respectively.

Then by Lemma A,

$$\begin{aligned}
\frac{2\xi_1}{R}\sin\frac{\beta+\psi-\gamma-\phi}{2}&=-\sin\frac{-\beta+\gamma+\phi+\psi}{2}-\sin\frac{\beta+\gamma+\phi-\psi}{2}\\
&\qquad+\sin\frac{\beta-\gamma+\phi+\psi}{2}-\sin\frac{\beta+\gamma-\phi+\psi}{2}
\end{aligned}$$

etc.; hence, after some manipulation,

$$\begin{aligned}
&\frac{2(\xi_2-\xi_3)}{R}\sin\frac{\gamma+\theta-\alpha-\psi}{2}\sin\frac{\alpha+\phi-\beta-\theta}{2}\\
&=-\sin\frac{\alpha-\theta}{2}\left(\sin\frac{-\alpha+\beta+\gamma+\theta+\phi-\psi}{2}-\sin\frac{-\alpha+\beta+\gamma+\theta-\phi+\psi}{2}\right.\\
&\qquad+\sin\frac{\alpha-\beta+\gamma-\theta+\phi+\psi}{2}-\sin\frac{\alpha-\beta+\gamma+\theta+\phi-\psi}{2}\\
&\qquad\left.+\sin\frac{\alpha+\beta-\gamma+\theta-\phi+\psi}{2}-\sin\frac{\alpha+\beta-\gamma-\theta+\phi+\psi}{2}\right)
\end{aligned}$$

$$= -\sin\frac{\alpha-\theta}{2}\,S(\alpha, \beta, \gamma;\ \theta, \phi, \psi), \text{ say.}$$

We observe that $S(\alpha, \beta, \gamma;\ \theta, \phi, \psi)$ is unchanged by simultaneous cyclic permutations of α, β, γ and θ, ϕ, ψ.

A corresponding calculation leads to the relation

$$\frac{2(\eta_2-\eta_3)}{R}\sin\frac{\gamma+\theta-\alpha-\psi}{2}\sin\frac{\alpha+\phi-\beta-\theta}{2}$$
$$= +\sin\frac{\alpha-\beta}{2}\,C(\alpha, \beta, \gamma;\ \theta, \phi, \psi)$$

where $C(\alpha, \beta, \gamma;\ \theta, \phi, \psi)$ is of the same form as $S(\alpha, \beta, \gamma;\ \theta, \phi, \psi)$ with sines replaced by cosines throughout. Hence

$$\frac{\eta_2-\eta_3}{\xi_2-\xi_3} = -\frac{C(\alpha, \beta, \gamma;\ \theta, \phi, \psi)}{S(\alpha, \beta, \gamma;\ \theta, \phi, \psi)} = \frac{\eta_3-\eta_1}{\xi_3-\xi_1} = \frac{\eta_1-\eta_2}{\xi_1-\xi_2} \quad \text{similarly.}$$

Hence (ξ_1, η_1), (ξ_2, η_2) and (ξ_3, η_3) are collinear, and Pascal's Theorem is proved.

8. We may now proceed further and establish the equation of the Pascal line; it is

$$y-\eta_1 = \frac{C(\alpha, \beta, \gamma;\ \theta, \phi, \psi)}{S(\alpha, \beta, \gamma;\ \theta, \phi, \psi)}(x-\xi_1)$$

or

$$C(\alpha, \beta, \gamma;\ \theta, \phi, \psi)\,.\,x + S(\alpha, \beta, \gamma;\ \theta, \phi, \psi)\,.\,y$$
$$= C(a, \beta, \gamma;\ \theta, \phi, \psi)\,.\,\xi_1 + S(\alpha, \beta, \gamma;\ \theta, \phi, \psi)\,.\,\eta_1$$

The reduction of the right-hand side to symmetrical form is achieved by multiplying out the two terms and combining corresponding terms by the sine addition formula. This yields the following expression, after further conventional manipulation:

$$-R\Big(\cos\frac{-\alpha+\beta+\gamma-\theta-\phi+\psi}{2} - \cos\frac{-\alpha+\beta+\gamma-\theta+\phi-\psi}{2}$$
$$+\cos\frac{\alpha-\beta+\gamma+\theta-\phi-\psi}{2} - \cos\frac{\alpha-\beta+\gamma-\theta-\phi+\psi}{2}$$
$$+\cos\frac{\alpha+\beta-\gamma-\theta+\phi-\psi}{2} - \cos\frac{\alpha+\beta-\gamma+\theta-\phi-\psi}{2}\Big)$$
$$= -R\,.\,C(\alpha, \beta, \gamma;\ -\theta,\ -\phi,\ -\psi)$$

The complete equation for the Pascal line is thus

$$x\,.\,C(\alpha, \beta, \gamma;\ \theta, \phi, \psi) + y\,.\,S(\alpha, \beta, \gamma;\ \theta, \phi, \psi)$$
$$= -R\,.\,C(\alpha, \beta, \gamma;\ -\theta,\ -\phi,\ -\psi).$$

The analogy between the form of this equation and that of a chord of the circle $x^2+y^2=R^2$ is striking rather than immediately

profitable, but provokes consideration of the manipulative properties (if any) of the functions C and S.

9. We begin with some obvious relations, mostly related closely to the geometry of the figure.

9.1. As already indicated

$$\begin{aligned}C(\alpha,\beta,\gamma;\ \theta,\phi,\psi)=C(\beta,\gamma,\alpha;\ \phi,\psi,\theta)&=C(\gamma,\alpha,\beta;\ \psi,\theta,\phi)\\ &=-C(\theta,\phi,\psi;\ \alpha,\beta,\gamma) \text{ etc., in turn}\\ &=+C(-\alpha,\ -\beta,\ -\gamma;\ -\theta,\ -\phi,\ -\psi), \text{ etc.}\end{aligned}$$

The same relations are true for the S's with a negative sign in the third line as well as the second.

9.2. A rather less obvious result, which, however, requires no more than enumeration of the 18 terms involved for immediate verification, is that

$$C(\alpha,\beta,\gamma;\ \theta,\phi,\psi)+C(\alpha,\beta,\gamma;\ \phi,\psi,\theta)+C(\alpha,\beta,\gamma;\ \psi,\theta,\phi)=0.$$

The same is true of the corresponding S's.

9.3. A still more subtle relation, but just as easy to verify, is that

$$C(\alpha,\beta,\gamma;\ \theta,\phi,\psi)+C(\theta,\gamma,\alpha;\ \psi,\beta,\phi)+C(\psi,\alpha,\theta;\ \phi,\gamma,\beta)=0$$

and it is also true that

$$\begin{aligned}C(\alpha,\beta,\gamma;\ -\theta,\ -\phi,\ -\psi)+C(\theta,\gamma,\alpha;\ -\psi,\ -\beta,\ -\phi)&\\ +C(\psi,\alpha,\theta;\ -\phi,\ -\gamma,\ -\beta)&=0.\end{aligned}$$

9.4 The relations of 9.2 and 9.3 lead to some further considerations.

10. From the results of 9.2, it follows at once that the Pascal lines of the three hexagons $(\alpha,\beta,\gamma;\ \theta,\phi,\psi)$, $(\alpha,\beta,\gamma;\ \phi,\psi,\theta)$, and $(\alpha,\beta,\gamma;\ \psi,\theta,\phi)$ are concurrent since the addition of the equations of any two of them yields that of the third. The same is true of the three hexagons

$$(\alpha,\beta,\gamma;\ \theta,\psi,\phi),\ (\alpha,\beta,\gamma;\ \psi,\phi,\theta),\quad\text{and}\quad(\alpha,\beta,\gamma;\ \phi,\theta,\psi).$$

These six hexagons are all that can be formed with the triads α,β,γ and θ,ϕ,ψ occurring alternately in the order of the vertices.

The six points can be divided into two triads in ${}^6C_3/2$ or 10 ways, and with each such division are associated two points of concurrence of three Pascal lines. There are thus twenty such points in all, known as the Steiner points of the 6-point. They are in general distinct, but no attempt is made to prove this here; the pairs of Steiner points are also known to be conjugate with respect to the circle, and all 20 lie in 4's on 15 straight lines, three through each point (H. F. Baker, *Introduction to Plane Geometry*, C.U.P., 1943, p. 349, Note 3: "On Pascal Lines").

11. The relations of 9.3 introduce us to the other sets of concurrencies among the Pascal lines. It has so far been convenient to consider each hexagon in terms of its triads of alternate vertices, but for this purpose it is more convenient to take the vertices in their natural order. Thus our original hexagon $(\alpha, \beta, \gamma;\ \theta, \phi, \psi)$ becomes $\alpha\,\phi\,\gamma\,\theta\,\beta\,\psi$ and we shall use this unpunctuated form to indicate that the vertices are in their natural order. The other hexagons involved in the relations of 9.3 become $\theta\,\beta\,\alpha\,\psi\,\gamma\,\phi$ and $\psi\,\gamma\,\theta\,\phi\,\alpha\,\beta$.

The relation between the order of vertices is not obvious in this literal form, but becomes so if we replace $\alpha, \phi, \gamma, \theta, \beta, \psi$ by the numbers 1 to 6 respectively. The three hexagons then are

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 6 & 3 & 2 \\ 6 & 3 & 4 & 2 & 1 & 5 \end{array}$$

and it can be seen that the same permutation of the order which changes the first into the second also changes the second into the third. But if we apply it yet again, the result is 2 1 6 5 4 3, which is the original hexagon. However, the relations of 9.3 show at once that the Pascal lines of these three hexagons are concurrent.

We can derive further concurrencies by moving round the vertices into the order

$$2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 1$$

from which the same permutation as before yields

$$5\ \ 6\ \ 2\ \ 1\ \ 4\ \ 3 \quad \text{and} \quad 1\ \ 4\ \ 5\ \ 3\ \ 2\ \ 6, \text{ and to}$$

$$3\ \ 4\ \ 5\ \ 6\ \ 1\ \ 2$$

from which we get

$$6\ \ 1\ \ 3\ \ 2\ \ 5\ \ 4 \quad \text{and} \quad 2\ \ 5\ \ 6\ \ 4\ \ 3\ \ 1.$$

Any further cyclic movement results in a repetition of previous groups.

Thus the original Pascal line is a member of three new groups of three concurrent lines, and if we write the seven hexagons down in the order in which we have obtained them they are

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 6 & 3 & 2 \\ 6 & 3 & 4 & 2 & 1 & 5 \\ 5 & 6 & 2 & 1 & 4 & 3 \\ 1 & 4 & 5 & 3 & 2 & 6 \\ 6 & 1 & 3 & 2 & 5 & 4 \\ 2 & 5 & 6 & 4 & 3 & 1 \end{array}$$

Denoting these by H_1 to H_7 in order, and their Pascal lines by

p_1 to p_7, we have so far proved that p_1, p_2, p_3; p_1, p_4, p_5; p_1, p_6, p_7 are concurrent sets.

Now taking an arbitrary cyclic movement of any hexagon other than H_1, we convert, say, H_2 into

$$5 \quad 1 \quad 6 \quad 3 \quad 2 \quad 4.$$

The usual permutation from this gives

	3 2 5 4 6 1, which is H_6	
and	4 6 3 1 5 2, which is a new hexagon H_8.	
Making H_2 into	1 6 3 2 4 5	
we derive	2 4 1 5 3 6	(H_9)
and	5 3 2 6 1 4	(H_5).

Thus p_2, p_6, p_8; p_2, p_5, p_9 are further concurrent triads.

Finally by writing

	$H_5 = 4$ 5 3 2 6 1	
we obtain	2 6 4 1 3 5	(H_{10})
and	1 3 2 5 4 6	(H_6)

whence p_5, p_6, p_{10} are concurrent. Further manipulation reveals

(a) that p_3, p_4, p_9; p_3, p_7, p_8; p_4, p_7, p_{10}; p_8, p_9, p_{10} are further concurrent triads, and

(b) that this is now a "closed shop", from which no further hexagons can be derived by the permutation

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \rightarrow 4 \quad 5 \quad 1 \quad 6 \quad 3 \quad 2$$

combined with cyclic changes of order, and in which no further concurrencies can be found.

In fact the whole figure is that of Desargues' Theorem, 10 lines meeting in threes in 10 points, three on each line.

The points of concurrence are called Kirkman points, and we may describe the ten hexagons involved as a Kirkman decad. Examination of the order of vertices of H_1 to H_{10} reveals that only one, namely H_1, has 1, 3, 5; 2, 4, 6 as alternate triads of vertices. Hence no two members of the same Kirkman decad can occur in the same pair of Steiner triads. Thus the hexagon $1\,x\,3\,y\,5\,z$, where $x\,y\,z$ is any permutation of 2 4 6, may be taken as the starting point of a fresh Kirkman decad, which will have no members in common with the first.

Thus the 60 hexagons derivable from the 6-point divide into 6 Kirkman decads, yielding a total of 60 Kirkman points, each decad contributing one member to each of the ten pairs of Steiner triads (H. F. Baker, loc. cit.).

12. To determine the coordinates of the Steiner point associated with the hexagon $\{\alpha, \beta, \gamma;\ \theta, \phi, \psi\}$ and those obtained from it by cyclic permutation of θ, ϕ, ψ, we have to solve the equations

$$xC(\alpha,\beta,\gamma;\ \theta,\phi,\psi)+yS(\alpha,\beta,\gamma;\ \theta,\phi,\psi)=-RC(\alpha,\beta,\gamma;\ -\theta,\ -\phi,\ -\psi)$$
$$xC(\alpha,\beta,\gamma;\ \phi,\psi,\theta)+yS(\alpha,\beta,\gamma;\ \phi,\psi,\theta)=-RC(\alpha,\beta,\gamma;\ -\phi,\ -\psi,\ -\theta)$$

For the nonce we shall write these for brevity as

$$xC+yS=-RC_-$$
$$xC'+yS'=-RC'_-$$

whence
$$x_s=-\frac{SC'_- - S'C_-}{SC'-S'C}$$

and
$$y_s=\frac{CC'_- - C'C_-}{SC'-S'C}$$

The problem is, of course, to find some method of dealing with these complex expressions. To begin with we reduce each of the C's and S' to three terms by combining the pairs of terms in which α, β, and γ have the negative sign. Thus

$$\tfrac{1}{2}C=-\sin\frac{-\alpha+\beta+\gamma+\theta}{2}\sin\frac{\phi-\psi}{2}-\sin\frac{\alpha-\beta+\gamma+\phi}{2}\sin\frac{\psi-\theta}{2}-\sin\frac{\alpha+\beta-\gamma+\psi}{2}\sin\frac{\theta-\phi}{2}$$

$$\tfrac{1}{2}S=+\cos\frac{-\alpha+\beta+\gamma+\theta}{2}\sin\frac{\phi-\psi}{2}+\cos\frac{\alpha-\beta+\gamma+\phi}{2}\sin\frac{\psi-\theta}{2}+\cos\frac{\alpha+\beta-\gamma+\psi}{2}\sin\frac{\theta-\phi}{2}$$

$$\tfrac{1}{2}C'=-\sin\frac{\alpha+\beta-\gamma+\theta}{2}\sin\frac{\phi-\psi}{2}-\sin\frac{-\alpha+\beta+\gamma+\phi}{2}\sin\frac{\psi-\theta}{2}-\sin\frac{\alpha-\beta+\gamma+\psi}{2}\sin\frac{\theta-\phi}{2}$$

$$\tfrac{1}{2}S'=+\cos\frac{\alpha+\beta-\gamma+\theta}{2}\sin\frac{\phi-\psi}{2}+\cos\frac{-\alpha+\beta+\gamma+\phi}{2}\sin\frac{\psi-\theta}{2}+\cos\frac{\alpha-\beta+\gamma+\psi}{2}\sin\frac{\theta-\phi}{2}$$

Reductions based on these and similar expressions yield

$$x_s=+R\,\frac{\Pi\sin\frac{\beta-\gamma}{2}\sum\cos\theta\sin^2\frac{\phi-\psi}{2}-\Pi\sin\frac{\phi-\psi}{2}\sum\cos\alpha\sin^2\frac{\beta-\gamma}{2}}{\Pi\sin\frac{\beta-\gamma}{2}\sum\sin^2\frac{\phi-\psi}{2}-\Pi\sin\frac{\phi-\psi}{2}\sum\sin^2\frac{\beta-\gamma}{2}}.$$

The easiest way to obtain y_s is merely to replace α by $\frac{\pi}{2}-\alpha$, etc., throughout x_s, and this gives

p_1 to p_7, we have so far proved that p_1, p_2, p_3; p_1, p_4, p_5; p_1, p_6, p_7 are concurrent sets.

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we derive	2	4	1	5	3	6	(H_9)
and	5	3	2	6	1	4	(H_5).

Thus p_2, p_6, p_8; p_2, p_5, p_9 are further concurrent triads.

Finally by writing

H_5=	4	5	3	2	6	1	
we obtain	2	6	4	1	3	5	(H_{10})
and	1	3	2	5	4	6	(H_6)

whence p_5, p_6, p_{10} are concurrent. Further manipulation reveals

(a) that p_3, p_4, p_8; p_3, p_7, p_8; p_4, p_7, p_{10}; p_8, p_9, p_{10} are further concurrent triads, and

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The problem is, of course, to find some method of dealing with these complex expressions. To begin with we reduce each of the C's and S' to three terms by combining the pairs of terms in which α, β, and γ have the negative sign. Thus

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$$\tfrac{1}{2}S=+\cos\frac{-\alpha+\beta+\gamma+\theta}{2}\sin\frac{\phi-\psi}{2}+\cos\frac{\alpha-\beta+\gamma+\phi}{2}\sin\frac{\psi-\theta}{2}+\cos\frac{\alpha+\beta-\gamma+\psi}{2}\sin\frac{\theta-\phi}{2}$$

$$\tfrac{1}{2}C'=-\sin\frac{\alpha+\beta-\gamma+\theta}{2}\sin\frac{\phi-\psi}{2}-\sin\frac{-\alpha+\beta+\gamma+\phi}{2}\sin\frac{\psi-\theta}{2}-\sin\frac{\alpha-\beta+\gamma+\psi}{2}\sin\frac{\theta-\phi}{2}$$

$$\tfrac{1}{2}S'=+\cos\frac{\alpha+\beta-\gamma+\theta}{2}\sin\frac{\phi-\psi}{2}+\cos\frac{-\alpha+\beta+\gamma+\phi}{2}\sin\frac{\psi-\theta}{2}+\cos\frac{\alpha-\beta+\gamma+\psi}{2}\sin\frac{\theta-\phi}{2}$$

Reductions based on these and similar expressions yield

$$x_s=+R\frac{\Pi\sin\frac{\beta-\gamma}{2}\sum\cos\theta\sin^2\frac{\phi-\psi}{2}-\Pi\sin\frac{\phi-\psi}{2}\sum\cos\alpha\sin^2\frac{\beta-\gamma}{2}}{\Pi\sin\frac{\beta-\gamma}{2}\sum\sin^2\frac{\phi-\psi}{2}-\Pi\sin\frac{\phi-\psi}{2}\sum\sin^2\frac{\beta-\gamma}{2}}.$$

The easiest way to obtain y_s is merely to replace α by $\frac{\pi}{2}-\alpha$, etc., throughout x_s, and this gives

$$y_s = +R\frac{\Pi \sin\frac{\beta-\gamma}{2}\sum \sin\theta \sin^2\frac{\phi-\psi}{2} - \Pi \sin\frac{\phi-\psi}{2}\sum \sin\alpha \sin^2\frac{\beta-\gamma}{2}}{\Pi \sin\frac{\beta-\gamma}{2}\sum \sin^2\frac{\phi-\psi}{2} - \Pi \sin\frac{\phi-\psi}{2}\sum \sin^2\frac{\beta-\gamma}{2}}$$

Equally no further manipulation is necessary to find (x_s', y_s'), the Steiner point of the associated triad, beyond interchanging, say, ϕ and ψ. Thus

$$x_s' = +R\frac{\Pi \sin\frac{\beta-\gamma}{2}\sum \cos\theta \sin^2\frac{\phi-\psi}{2} + \Pi \sin\frac{\phi-\psi}{2}\sum \cos\alpha \sin^2\frac{\beta-\gamma}{2}}{\Pi \sin\frac{\beta-\gamma}{2}\sum \sin^2\frac{\phi-\psi}{2} + \Pi \sin\frac{\phi-\psi}{2}\sum \sin^2\frac{\beta-\gamma}{2}}$$

and

$$y_s' = +R\frac{\Pi \sin\frac{\beta-\gamma}{2}\sum \sin\theta \sin^2\frac{\phi-\psi}{2} + \Pi \sin\frac{\phi-\psi}{2}\sum \sin\alpha \sin^2\frac{\beta-\gamma}{2}}{\Pi \sin\frac{\beta-\gamma}{2}\sum \sin^2\frac{\phi-\psi}{2} + \Pi \sin\frac{\phi-\psi}{2}\sum \sin^2\frac{\beta-\gamma}{2}}$$

Poised for the kill!

Hence $$\frac{x_s x_s' + y_s y_s'}{R^2}$$

$$= \frac{\Pi \sin^2\frac{\beta-\gamma}{2}\left\{\left(\sum \cos\theta \sin^2\frac{\phi-\psi}{2}\right)^2 + \left(\sum \sin\theta \sin^2\frac{\phi-\psi}{2}\right)^2\right\} - \Pi \sin^2\frac{\phi-\psi}{2}\left\{\left(\sum \cos\alpha \sin^2\frac{\beta-\gamma}{2}\right)^2 + \left(\sum \sin\alpha \sin^2\frac{\beta-\gamma}{2}\right)^2\right\}}{\Pi \sin^2\frac{\beta-\gamma}{2}\left(\sum \sin^2\frac{\phi-\psi}{2}\right)^2 - \Pi \sin^2\frac{\phi-\psi}{2}\left(\sum \sin^2\frac{\beta-\gamma}{2}\right)^2}$$

Now $\left(\sum \cos\theta \sin^2\frac{\phi-\psi}{2}\right)^2 + \left(\sum \sin\theta \sin^2\frac{\phi-\psi}{2}\right)^2$

$$= \sum \sin^4\frac{\phi-\psi}{2} + 2\sum \sin^2\frac{\psi-\theta}{2}\sin^2\frac{\theta-\phi}{2}\cos(\phi-\psi)$$

$$= \sum \sin^4\frac{\phi-\psi}{2} + 2\sum \sin^2\frac{\psi-\theta}{2}\sin^2\frac{\theta-\phi}{2} - 12\Pi \sin^2\frac{\phi-\psi}{2}$$

$$= \left(\sum \sin^2\frac{\phi-\psi}{2}\right)^2 - 12\Pi \sin^2\frac{\phi-\psi}{2}.$$

Treating $\left(\sum \cos\alpha \sin^2\frac{\beta-\gamma}{2}\right)^2 + \left(\sum \sin\alpha \sin^2\frac{\beta-\gamma}{2}\right)^2$ similarly we have at once that

$$x_s x_s' + y_s y_s' = R^2$$

so that the Steiner points of these associated triads are conjugate with respect to the circle.

There is a concise account of the properties of the complete Pascal figure in a note at the end of Salmon's Conic Sections. R. S.

TWO PROBLEMS ON IMPULSIVE MOTION

By J. O'Keeffe

1. There are two types of problem on impulsive motion which receive a somewhat cursory treatment in most of the standard text books on dynamics. The first, and more elementary, is as follows: Two rods AB and BC are jointed at B and an impulse $\mathbf{J}$ is applied to one of them at a certain point. Find the motion. As one of the steps in the solution of this problem we are usually recommended to take moments about the point B. But, we may remark, B is neither a fixed point nor the centre of gravity of the system. How then are we justified in taking moments about this point? This question will be considered in paragraph 2 below.

The second is the problem of a general framework of rods which receives a given blow $\mathbf{J}$ at a given point, to be solved by means of Bertrand's theorem. The method of solution suggested is as follows: Find the velocities of the centres of gravity of the rods in terms of the velocity of the point of application of the impulse and the angular velocities of the rods; write down the equations of linear momentum for the system as a whole; write down the kinetic energy of the system, and then the conditions that this kinetic energy shall have a stationary value with respect to variations in the angular velocities of the rods; we then have a set of linear equations for the unknown quantities. Two questions immediately suggest themselves: Firstly, why do we consider variations in the angular velocities only? Secondly, in taking the velocities of the point of application as invariable are we not using Kelvin's theorem rather than Bertrand's and thus solving what is essentially a maximum-value problem by means of a minimum principle (the error not affecting the answer for the reason that only first derivatives need be considered in either method)? These matters will be considered in paragraph 3. It is possible, of course, to avoid this difficulty altogether by treating such problems by means of Lagrange's equations as is done in the most recent of the references quoted.

2. The differential equation for the angular momentum of a system about a moving point O' takes, in the case of finite forces, the form

$$\frac{d\mathbf{H}(O')}{dt}+\mathbf{V}(O')\wedge\mathbf{L}=\mathbf{\Gamma}(O'), \tag{1}$$

where $\mathbf{H}(O')$ is the angular momentum of the system about O', $\mathbf{L}$ is the linear momentum of the system, $\mathbf{V}(O')$ is the velocity of the moving point O' and $\mathbf{\Gamma}\ (O')$ is the sum of the moments about O' of the external forces acting on the system. The corresponding equation for impulsive forces will be obtained by integrating (1)

over a small time-interval (t_0, t_1) and neglecting changes of position but taking account of changes in velocity and momentum.

We obtain

$$\Big[\mathbf{H}(O')\Big]_{t_0}^{t_1} + \int_{t_0}^{t_1} \mathbf{V}(O') \wedge \mathbf{L}\,dt = \int_{t_0}^{t} \mathbf{\Gamma}(O')\,dt = \mathbf{X}(O'), \qquad (2)$$

say. We thus see that the process of taking moments about a moving joint B is valid only in cases where the second term of equation (2) vanishes; the most important of which is where the initial velocity of B is in the same direction as that of the centre of gravity. This is the case in reference 1.

3. In the Bertrand problem the first difficulty relates essentially to the question of how the given impulse **J** is to be introduced into our calculations. The condition usually stated for Bertrand's theorem is that the constraints shall be of a kind which do no work in a small displacement, but this condition is too broad for the type of application we are considering. It is necessary to have an equation introducing the given quantity **J** and this necessity imposes a further limitation on the types of constraint permissible: constraints involving external reactions are not allowed; for these, even though they do no work, will always interfere with the equation by means of which we take account of **J**. The upshot of the matter is that we are restricted to variations which could be caused by internal constraints, i.e., to variations in the angular velocities.

The easiest way to take account of **J** is to regard it as specifying the velocity of the centre of gravity of the system, and to take this velocity as a given constant vector. It is easily determined from the equation for the linear motion of the centre of gravity. Next write down the sum of the kinetic energies of the rods in terms of this constant vector and the angular velocities of the rods, and express that this kinetic energy shall be a maximum with respect to variations in these angular velocities. This will give a solution of the problem.

This is a different matter, however, from taking the velocity of the point of application of the impulse to be a given constant, as is sometimes done. This, in the opinion of the present writer, is an erroneous procedure, even though it gives the same answer. It amounts to using a minimum principle to solve a maximum value problem.

J. O'K.

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A FUNCTIONAL EQUATION CHARACTERISING THE SINE

By R. A. Rosenbaum and S. L. Segal

Recent articles on equations characterising the trigonometric functions ([1], [2]) prompt a consideration of still another equation, which might be classified under the heading of " Students' Mathematical Mythology ". In [3], Mr. Heafford multiplies the " obvious " $\sin(x+y) = \sin x + \sin y$ by the equally " clear " $\sin(x-y) = \sin x - \sin y$ to obtain the (correct!) result, $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$. Having independently observed this curious relation, we were led to study the functional equation,

$$f(x+y)f(x-y) = f^2(x) - f^2(y) \tag{1}$$

where f is a real-valued function, defined over all real numbers.

It is clear that (1) has no unique non-trivial solution, for the additive function, $f(x) = kx$ satisfies it; and if no conditions of continuity are imposed the everywhere discontinuous additive functions created by using a Hamel basis for real numbers are also solutions.

However, rather modest restrictions lead to a satisfying result, even if (1) is generalized so that f is a complex-valued function defined over all complex numbers. In this paper we shall demonstrate the following theorem:

Theorem 1. If $f(z)$ is a complex-valued function defined over all complex numbers, continuous at a point, bounded on every closed set, and such that the set of non-zero zeros of f is either empty or bounded away from zero, which satisfies the functional equation:

$$f(z_1+z_2)f(z_1-z_2) = f^2(z_1) - f^2(z_2) \tag{2}$$

for all complex z_1, z_2, then either $f(z) = k_1 z$, or $f(z) = k_2 \sinh k_3 z$, where k_1, k_2, k_3 are arbitrary complex constants (provided that $\lim_{z\to 0}\lim_{n\to\infty} \frac{n}{z} f\left(\frac{z}{n}\right)$ exists). We take $\sinh z$ as defined by its series.

An obvious analogue of this result is that if f is restricted to be a real-valued function of a real variable and the other conditions of Theorem 1 are satisfied, then the only solutions of (1) are $c_1 x$, $c_2 \sin c_3 x$ and $c_4 \sinh c_5 x$, where the c_i are real constants. As Theorem 2 of this paper we shall show that for such an f, the condition on the non-zero zeros and the limit condition may be dropped.

We hereafter neglect the trivial solution $f(z) \equiv 0$ of (2).

The advantage in (2) as a functional equation characterising the sine (hyperbolic sine) lies in that the desired result can be derived without the imposition of auxiliary restrictions based on a previous knowledge that the sine (hyperbolic sine) is a solution, and that only

one equation rather than a system of equations is used. Indeed from one point of view (1) and (2) may be regarded as generalizations of the Cauchy functional equation $F(x+y)=F(x)+F(y)$ and the solutions of (2) as the " simplest " odd functions.

We now outline a proof of Theorem 1. We observe that under the conditions of the theorem:

I. $f(0)=0$.

Proof. Set $z_1=z_2=0$ in (2).

II. f is an odd function.

Proof. Set $z_1=0$ in (2) and use I. If $f(z_2)=0$, note that replacing z_2 by $-z_2$ in (2) and comparing the two equations gives $f^2(z_2)=f^2(-z_2)$.

III. If f is continuous at a point other than 0, then f is continuous at 0.

Proof. In (2) let z_1 be the non-zero point at which f is assumed to be continuous; take limits as $z_2 \to 0$ and use I.

IV. If f is continuous at 0, then f is everywhere continuous.

Proof. We make use of the following three equations, all of which follow directly from (2) for any complex z and h:

$$f(2z)f(2h)=f^2(z+h)-f^2(z-h) \tag{3}$$

$$f^2(z+h)f^2(z-h)=[f^2(z)-f^2(h)]^2 \tag{4}$$

$$f(z)f(z+2h)=f^2(z+h)-f^2(h) \tag{5}$$

We eliminate $f^2(z-h)$ between (3) and (4) and solve the resulting quadratic in $f^2(z+h)$ which has the solutions:

$$f^2(z+h)=\frac{f(2z)f(2h)\pm[f^2(2z)f^2(2h)+4(f^2(z)-f^2(h))^2]^{\frac{1}{2}}}{2}$$

where either the plus or minus sign may hold. Taking square roots of both sides and limits as $h\to 0$, we note that the limit on the right exists because of III, and we obtain the following four possibilities for the limit on the left:

$$\lim_{h\to 0} f(z+h) \quad \text{is either} \quad f(z),\ -f(z),\ i\,f(z), \text{ or } -i\,f(z)$$

for a particular z. Suppose any of the last three possibilities holds, then taking limits as $h\to 0$ in (5) we see that $f(z)$ must equal 0 in these cases. Hence in all cases $\lim_{h\to 0} f(z+h)=f(z)$.

V. $f(z+2\alpha)=f(z)$ if and only if $f(\alpha)=0$, $(\alpha\neq 0)$.

Proof. " if " We use the following equations derived from (2) and valid for any non-zero α of $f(z)$ and any complex z.

$$f(z+\alpha)f(z-\alpha)=f^2(z)-f^2(\alpha)=f^2(z) \tag{6}$$

$$f^2(z+\alpha)-f^2(z)=f(2z+\alpha)f(\alpha)=0 \tag{7}$$

Eliminating $f^2(z)$ between (6) and (7) we see that if $f(z+\alpha)\neq 0$, then $f(z+\alpha)=f(z-\alpha)$, whereas if $f(z+\alpha)=0$, then it follows from (7) that $f(z+\alpha)=f(z)=f(z-\alpha)=0$.

"only if" We use the following equations derived from (2) and valid for any "period" 2α of f, and any complex z.

(8) $$f^2(z+\alpha)-f^2(\alpha)=f(z)f(z+2\alpha)=f^2(z)$$

(9) $$f^2(z+\alpha)-f^2(z)=f(2z+\alpha)f(\alpha).$$

Eliminating $f^2(z+\alpha)-f^2(z)$ between (8) and (9) we see that if $f(\alpha)\neq 0$, then $f(2z+\alpha)$ (and hence $f(z)$) is identically constant, and hence from (2) identically 0. Therefore $f(\alpha)=0$.

VI. For α as in V, $f(z+\alpha)=$ either $f(z)$ or $-f(z)$ according as $f(\frac{1}{2}\alpha)=0$ or $\neq 0$.

Proof. This follows directly from (7) and V.

VII. There exist simply connected regions R, S, and T, $T\subset S\subset R$, with the following properties: R, S, and T all contain the origin, but no other zeros of f; if $z_1, z_2\in T$, then z_1+z_2, $z_1-z_2\in S$; $f(T)\subset R$, $f(S)\subset R$. For R, S and T so defined, the mapping $T\to f(T)$ is one-to-one.

Proof. The existence of R,S, and T follows from the hypothesis of the theorem that the non-zero zeros (if they exist) are bounded away from 0, the continuity of f, and the fact that $f(0)=0$. That $T\to f(T)$ is one-to-one may be proved as follows. Let z_1, z_2 be any two points of T such that $f(z_1)=f(z_2)$. Then it follows from (2) that

(10) $$f(z_1+z_2)f(z_1-z_2)=0$$

Since f is a mapping from the complex plane onto the complex plane, at least one of the factors on the left of (10) must be 0. But by hypothesis both factors are members of R, and hence either $z_1=z_2$, or $z_1=-z_2$. In the latter case, since f is odd, $f(z_1)=-f(z_2)=-f(z_1)$. And therefore since $z_1\in T$, $z_1=0$. Hence $f(T)\to T$ is one-to-one.

VIII. f is differentiable at 0.

Proof. Let R, S, and T be defined as in VII, and let C be an open disk centered at 0 such that $C\subset T$. Then it is clear from VII that f maps C onto $f(C)$ in a one-to-one fashion. We now make use of the following equation derived from (2) and valid for all complex h and all positive integers ν.

(11) $$f([2\nu+1]h)f(h)=f^2([\nu+1]h)-f^2(\nu\,.\,h)$$

Summing both sides of (11) from $\nu=1$ to n, we obtain:

(12) $$f(h)\sum_{\nu=1}^{n} f([2\nu+1]h)=f^2([n+1]h)-f^2(h)$$

for all positive integers n. Setting $h = B/n$ where B is any non-zero complex number $\in C$, we may rewrite (12) as:

$$\frac{f(B/n)}{B/n}\sum_{\nu=0}^{n} f([2\nu+1]B/n)2B/n = 2f^2(B+B/n)$$

and since f is everywhere continuous passage to the limit gives formally:

$$\lim_{n\to\infty}\frac{f(B/n)}{B/n}\int_0^{2B} f(z)\,dz = 2f^2(B) \tag{13}$$

Since the right side of (13) exists and $\neq 0$ (since $B\in C\subset T$, $B\neq 0$), so does the left and $\neq 0$. Since f is everywhere continuous the integral in (13) exists. Hence

$$\int_0^{2B} f(z)\,dz \neq 0 \text{ if and only if } \lim_{n\to\infty}\frac{f(B/n)}{B/n} \text{ exists, } (H\neq 0,\ B\in C) \tag{14}$$

and if the limit exists it is not zero.

We now show that the assumption that a point $B_1\neq 0$, $B_1\in C$, exists such that $\int_0^{2B_1} f(z)\,dz = 0$ leads to a contradiction and hence the limit in question exists for arbitrary non-zero $B\in C$. We use the following equation derived from (2) and valid for all complex h and positive integers ν.

$$f(2\nu h)f(2h) = f^2([\nu+1]h) - f^2([\nu-1]h) \tag{15}$$

Arguing analogously to the previous paragraph we sum both sides from 1 to n, and this time setting $h = B/2n$, where B as before is any non-zero complex number $\in C$, we obtain

$$\frac{f(B/n)}{B/n}\sum_{\nu=1}^{n} f(\nu B/n)B/n = f^2([n+1]B/2n) + f^2(B/2) - f^2(B/2n)$$

Taking limits as $n\to\infty$, and arguing as before, we obtain

$$\lim_{n\to\infty}\frac{f(B/n)}{B/n}\int_0^{B} f(z)\,dz = 2\,f^2(B/2) \tag{16}$$

$$\int_0^{B} f(z)\,dz \neq 0 \text{ if and only if } \lim_{n\to\infty}\frac{f(B/n)}{B/n} \text{ exists, } (B\neq 0,\ B\in C) \tag{17}$$

Now for our hypothesized B_1 the integral of (14) vanishes, hence the limit does not exist; therefore the integral of (17) vanishes, and so the left sides of (13) and (16) are equal, whence the right sides are equal. Therefore if such a B_1 exists it must satisfy $f^2(B_1/2) = f^2(B_1)$. That is, either $f(B_1) = f(B_1/2)$, or $f(B_1) = -f(B_1/2) = f(-B_1/2)$. But by VII $C\to f(C)$ is one-to-one, and hence $B_1 = 0$ contrary to hypothesis. Hence $\lim_{n\to\infty}\frac{f(B/n)}{B/n}$ exists and $\neq 0$ for all $B\in C$, $B\neq 0$.

It remains to show that this limit is constant, from which it follows that $f'(0)$ exists. To this end we consider the function $G(B)$

defined for all $B \in C$, $B \neq 0$, by: $G(B) = \lim_{n\to\infty} n f(B/n)$; and for 0 by $G(0) = 0$. From (13) we immediately obtain the representation (since the integral is never 0 for $B \neq 0$): $G(B) = 2B f^2(B) \Big/ \int_0^{2B} f(z)\,dz$, $(B \neq 0, B \in C)$, from which it follows that $G(B)$ is continuous everywhere for $B \in C$, except possibly at 0. Furthermore, from (13), (16), and the argument of the preceding paragraph,

$$\lim_{n\to\infty} \frac{f(B/n)}{B/n} = \lim_{n\to\infty} \frac{f(B/2n)}{B/2n},$$

or in other words, $G(B) = 2\,G(B/2)$. We now consider the following equations derived from (2) and valid for all B_p, B_q:

$$f(B_p/n) f(B_q/n) = f^2\left(\frac{B_p + B_q}{2n}\right) - f^2\left(\frac{B_p - B_q}{2n}\right)$$

$$f\left(\frac{B_p + B_q}{n}\right) f\left(\frac{B_p - B_q}{n}\right) = f^2(B_p/n) - f^2(B_q/n)$$

These equations may be combined in two ways to give:

$$(18)\quad [f(B_p/n) + i\,f(B_q/n)]^2 = 2i\left[f^2\left(\frac{B_p + B_q}{2n}\right) - f^2\left(\frac{B_p - B_q}{2n}\right)\right] + f\left(\frac{B_p - B_q}{n}\right) f\left(\frac{B_p + B_q}{n}\right)$$

$$(19)\quad [f(B_p/n) - i\,f(B_q/n)]^2 = -2i\left[f^2\left(\frac{B_p + B_q}{2n}\right) - f^2\left(\frac{B_p - B_q}{2n}\right)\right] + f\left(\frac{B_p - B_q}{n}\right) f\left(\frac{B_p + B_q}{n}\right)$$

We consider first (18) and multiply both sides of this equation by $2n^2$, take limits as $n \to \infty$, and using the definition of $G(B)$ and the fact that $G(B) = 2\,G(B/2)$, obtain for all B_p, $B_q \neq 0$, $B_p^2 \neq B_q^2$:

$$2[G(B_p) + i\,G(B_q)]^2 = i[G^2(B_p + B_q) - G^2(B_p - B_q)] + 2\,G(B_p + B_q)\,G(B_p - B_q)$$

Multiplying both sides by i, and taking square roots gives

$$(20)\qquad \pm(i+1)[G(B_p) + i\,G(B_q)] = G(B_p - B_q) + i\,G(B_p + B_q)$$

Similarly we obtain from (19)

$$(21)\qquad \pm(i-1)[G(B_p) - i\,G(B_q)] = G(B_p - B_q) - i\,G(B_p + B_q)$$

(20) and (21) are simultaneously valid for all B_p, B_q for which B_p, $B_q \neq 0$, and $B_p \neq B_q$. Thusfar any of the four possible combinations of the signs of (20), (21): $(+, +)$; $(-, -)$; $(-, +)$; $(+, -)$ may hold; however the first three of these lead to contradictions and only the last is valid. We shall demonstrate how the case $(+, +)$ leads to a contradiction; the other cases may be treated analogously.

Suppose that in (20) and (21) both plus signs hold simultaneously.

Then adding the two equations gives: $i[G(B_p)+G(B_q)]=G(B_p-B_q)$ valid for all $B_p, B_q\neq 0, B_p^2\neq B_q^2$. Choosing $B_q=B_p/2$ in this equation and using the fact that $G(B_p)=2\,G(B_p/2)$ we obtain: $3i\,G(B_p/2)=G(B_p/2)$ which is impossible, since from its definition $G(B_p/2)\neq 0$.

Now with the only choice of signs for which (20) and (21) are valid, (namely, + in (20), − in (21)), we subtract (21) from (20) and obtain:

$$G(B_p)+G(B_q)=G(B_p+B_q) \tag{22}$$

Letting B_q approach 0 in (22) we see that since G is continuous for all non-zero $B\in C$, and $G(0)=0$ by definition, that G is also continuous at 0. Hence $G(B)$ is continuous and (22) holds for all $B\in C$. We may further note that though each function f satisfying (2) may perhaps lead to a different function G satisfying (22), to each f corresponds one and only one function G.

Equation (22) is the additive functional equation for a complex variable; however, from this we cannot immediately conclude, as we could were the variable real, that $G(B)=kB$, k an arbitrary constant, and hence from our previous remarks that $f'(0)$ exists.* However, certain other information which we have concerning G will allow us to obtain the desired conclusion.

We first note that by similar methods to those used for the real additive functional equation, one can prove that:

$$G(kB)=k\,G(B)\quad \text{for all real } k \text{ and complex } B\in C \tag{23}$$

(e.g., one first proves (23) for integral k by induction, then for rational k by a simple substitution, and finally for real k by a continuity argument). Furthermore with domain restricted to be C instead of the whole plane, we observe that G satisfies (2) and all the other hypotheses of Theorem 1, and hence by VII the mapping $C\to G(C)$ is one-to-one, and also the representation (16) holds with f replaced by G. This latter (using (23)) reduces to:

$$\int_0^B G(z)\,dz=\tfrac{1}{2}B\,G(B)\qquad B\neq 0, B\in C \tag{24}$$

Since G is continuous and one-to-one it has an inverse G^{-1} which is also one-to-one and which is easily seen to be not only continuous but additive as well. It follows that (24) holds with G replaced by G^{-1} and B by $G(B)$:

* The well-known fact that $F(x)=kx$ is the only continuous solution of the additive functional equation for a real variable was first proved by Cauchy. His work was extended and numerous different proofs were given by Darboux, Sierpinski, Ostrowski, and others. The best result known is due to Ostrowski who proved that the theorem is true when continuity is replaced by boundedness above on a set of positive measure (*Jrsbrch. d. Deut. Math. Verein.* *38*: 54–62, 1929). (Another proof is by H. Kestelman, *Fund. Math.* *34*, 144–7, 1947). We know of no literature on the complex case considered here. That continuity alone is insufficient for uniqueness is clear since, e.g., $Re(z)$ satisfies (22).

(25) $$\int_0^{G(B)} G^{-1}(z)\,dz = \tfrac{1}{2}\, G(B)G^{-1}(G(B)) = \tfrac{1}{2}\, B\, G(B)$$

Finally $\lim_{h\to 0} G(h)/h$ exists by hypothesis (where the limit is taken along any straight line through the origin) and from the additive nature of G follows:

(26) $$\lim_{h\to 0} \frac{G(w+h)-G(w)}{h} = \lim_{h\to 0} G(h)/h \quad w\in C$$

Now in (25) we fix B at any particular non-zero value and make the transformation of variable $z=G(w)$ and obtain:

$$\tfrac{1}{2}B\, G(B) = \lim_{h\to 0} G(h)/h \int_0^B w\,dw = \lim_{h\to 0} G(h)/h\ \tfrac{1}{2}B^2$$

that is: $\lim_{h\to 0} G(h)/h = G(B)/B$ is a constant independent of h, and $\neq 0$. It follows immediately that $G'(0)$ exists and that $G(B) = G'(0)B$ for all $B\in C$, whence $f'(0)$ exists.

IX. f is differentiable everywhere and $2f(z)f'(z) = f'(0)f(2z)$; if $f(z)=0$, then $f'(z)=f'(0)$ or $-f'(0)$ according as $f(\tfrac{1}{2}z)=0$ or $\neq 0$.

Proof. From (2) we have: $f(2z+h)f(h)/h = [f^2(z+h)-f^2(z)]/h$, and taking limits as $h\to 0$ gives the stated result. If $f(z)=0$, $z\neq 0$, then with α as in V and using the results of V and VI

$$\frac{f(\alpha+h)-f(\alpha)}{h} = \pm\frac{f(h)}{h}$$

and the stated result follows.

X. $f(z)=$ either $k_1 z$ or $k_2 \sinh k_3 z$, where k_1, k_2, k_3 are arbitrary complex constants.

Proof. f is regular in the whole plane and hence may be expanded in a Taylor series around 0. From the argument of VIII, $f'(0)\neq 0$. Setting $f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$ and comparing coefficients in the formula of IX, an easy induction shows that $a_{2\nu}=0$ for all non-negative integers ν. $a_0=f(0)=0$; the comparison of coefficients gives:

(27) $$2^{2\nu}a_{2\nu}a_1 = 2^{2\nu}a_{2\nu}f'(0) = \sum_{p=1}^{2\nu} 2p\, a_p a_{2\nu+1-p}$$

Suppose $a_{2\nu}=0$ for $\nu\leqslant k-1$, then for $\nu=k$ the right side of (27) reduces to $(4k+2)a_{2k}a_1$. Since $a_1=f'(0)\neq 0$, we obtain:

$$2^{2k-1}a_{2k} = (2k+1)a_{2k}$$

which is clearly possible for integral k only if $a_{2k}=0$. We have now shown that $f(z)$ is of the form $\sum_{\nu=0}^{\infty} a_{2\nu+1}z^{2\nu+1}$. (It would seem that an argument similar to the above might also be used to obtain a formula for the coefficients with odd subscripts $\geqslant 5$ in terms of a_1 and a_3,

however, this is considerably more complicated and it seems easier to transform (28) into some other form in order to consider the coefficients with odd subscripts. We wish to note that while, of course, there are many possible variants of the methods used here, our approach seems to be the directest and simplest in that it naively assumes no properties of sinh z other than the development in series set out in the theorem).

We use (2), IX, the regularity of f, and the assumption that it is not identically 0, to argue that there exists a simply connected region $\mathscr{R}$ of the complex plane in which the following transformation of IX holds. Differentiating in IX, we get:

$$f(z)f''(z)+f'(z)^2=f'(0)f'(2z)=[f'(0)]^2\frac{f(4z)f(2z)}{2f^2(2z)}$$
$$=[f'(0)]^2\frac{f^2(3z)f^2(z)-f^4(z)}{2f^2(2z)f^2(z)}=[f'(0)]^2\frac{[f^2(2z)-f^2(z)]^2-f^4(z)}{2f^2(2z)f^2(z)}$$
$$=[f'(0)]^2\frac{f^2(2z)-2f^2(z)}{2f^2(z)}=2[f'(z)]^2-[f'(0)]^2.$$

And so we obtain $f(z)f''(z)=[f'(z)]^2-[f'(0)]^2$

Differentiating this last relationship once more, we get:

$$(28) \qquad f(z)f^{(3)}(z)=f'(z)f''(z)$$

Since $f(z)\not\equiv 0$, there exist simply connected subregions of $\mathscr{R}$ in which $f(z)\neq 0$. If $f''(z)\equiv 0$ in such a subregion, then $f(z)=k_1z$ in this subregion and hence by the identity theorem for analytic functions, in the whole plane. If there is a subregion $\mathscr{R}_1$, in which $f''(z)\neq 0$, then dividing through in (28) by $f(z)f''(z)$ and integrating, we obtain:

$$(29) \qquad f''(z)=cf(z) \quad c\neq 0$$

Comparison of the Taylor series coefficients in (29) and the fact that the coefficients with even subscripts are all 0, show immediately that $f(z)=\sum_{\nu=0}^{\infty}k_2\frac{(k_3z)^{2\nu+1}}{(2\nu+1)!}=k_2\sinh k_3z$ in $\mathscr{R}_1$, and hence in the whole plane. Our central theorem is thus proved.

THEOREM 2. If $f(x)$ is a real-valued function defined over all real numbers, continuous at a point, bounded on every closed set, which satisfies the functional equation (1) for all real x, y, then $f(x)$ = either c_1x, $c_2\sin c_3x$, $c_4\sinh c_5x$, where the c_i are arbitrary real constants.

Proof. I through VI follow as for Theorem 1. The real variable analogue of (13) is then set up as in VIII. That there exists a real number B for which the integral is not 0 follows immediately from the assumption that $f(x)$ is not identically 0. The existence of $f'(0)$ then follows immediately without any further argument. The formulas of IX follow as there. The existence of higher order deriva-

tives is easily shown, and one can then prove that either $f''(x) \equiv 0$, or $f''(x) = cf(x)$, $c \neq 0$, by methods analogous to those of X. Solving these differential equations and applying $f(0) = 0$, we obtain the stated result.

We wish only to observe further that it would seem that the "simplest" even functions, namely $f(z) = C$, $C \neq 0$, and $f(z) = k_1 \cosh k_2 z$ $(k_3 \cos k_4 z)$, may be similarly characterized by the functional equation: $g(z_1 + z_2) g(z_1 - z_2) = g^2(z_1) + g^2(z_2) + K$, where K is an arbitrary non-zero complex constant, though we have not proved this.

R. A. R. and S. L. S.

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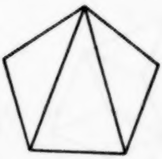
3. Phillip Heafford, *Mathematics for Fun* (Hutchinson), p. 49.

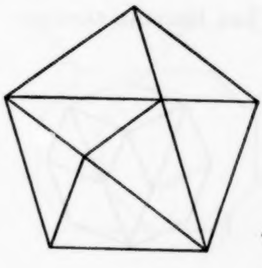
TILING PATTERNS FOR REGULAR POLYGONS

BY C. DUDLEY LANGFORD

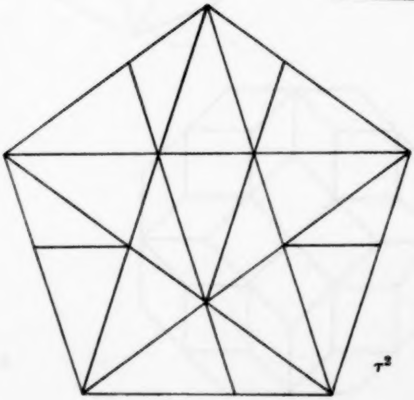
I. THE REGULAR PENTAGON



1

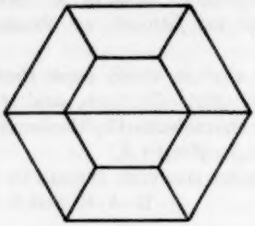


τ



τ^2 &c.

II. The Regular Hexagon



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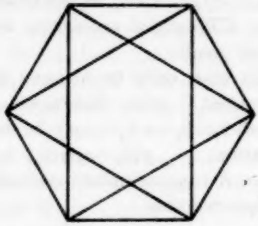

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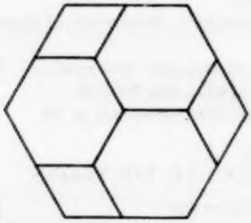

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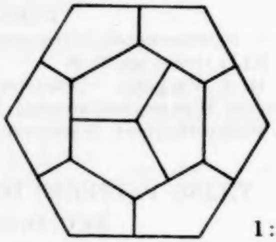
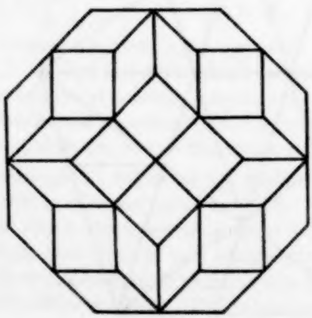


1 : $\sqrt{3}$

III. The Regular Octagon

1

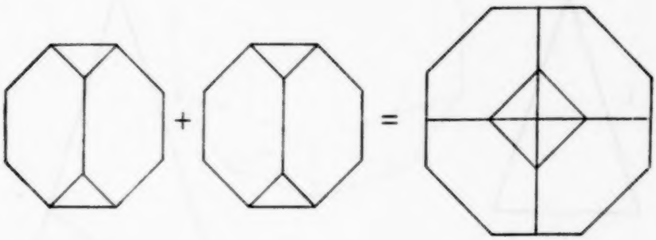
$\sqrt{2}$

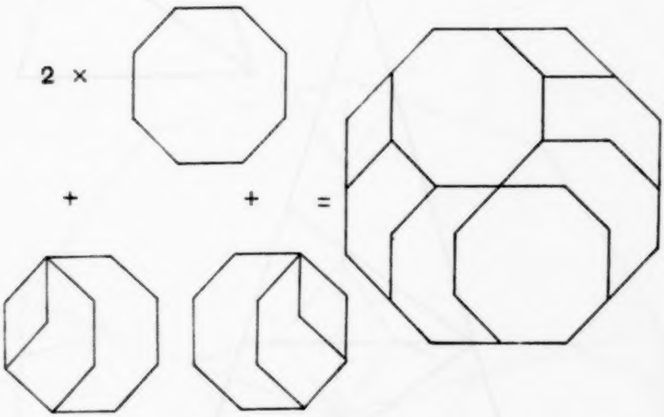


2

&c.

III. The Regular Octagon: dissection

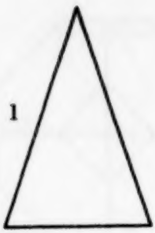




IV. The Regular Decagon

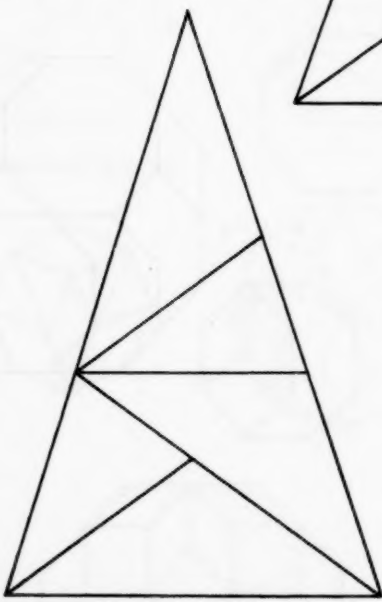
The same shapes of tile are used as for the regular pentagon (I). The diagrams show only one-tenth of the complete decagon.

1



τ

τ^2



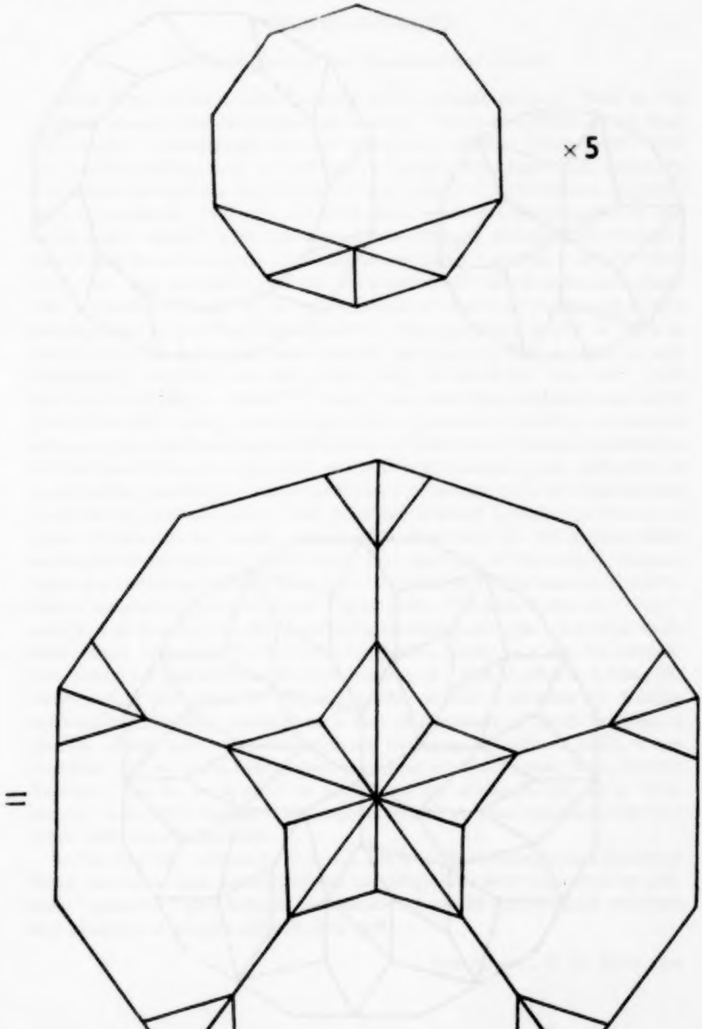
&c.

Note that considerations of area illustrate the identities

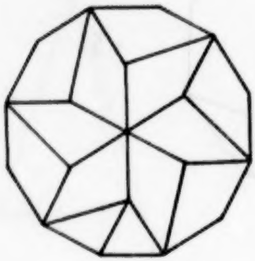
$$\tau^2 = \tau + 1, \ \tau^4 = 3\tau + 2, \ \tau^6 = 8\tau + 5, \ \ldots$$

which also indicate the numbers of each tile required for each polygon in the series. (The numbers are, of course, successive terms of Fibonacci's series; 1, 1, 2, 3, 5, 8. . .)

IV. The Regular Decagon: dissection



V. The Regular Dodecagon



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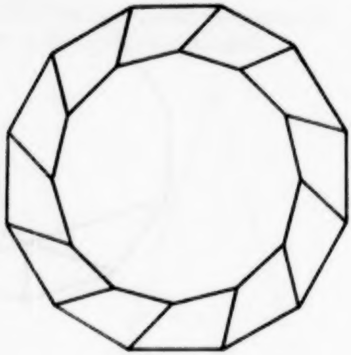
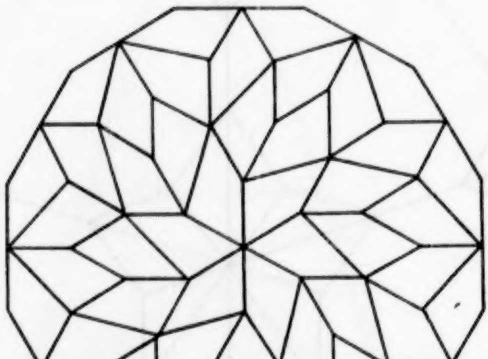


$\sqrt{2}$



2

&c.

C.D.L.

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

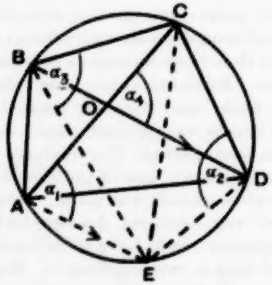
Dear Sir,—I have been looking with interest at note 2873 in the current issue of the *Mathematical Gazette*. The moral seems to be that the concept " inextensible string " contains a double idealisation: first the overt condition that the extension is negligibly small; plus secondly a concealed condition that no part of the energy of deformation is elastically recoverable. Though not all strings which are inextensible in the overt sense comply with the second condition, I should have thought that there would be little difficulty in devising " strings " which effectively do: but whatever may be the truth about this it does seem plain that if we are to continue to set problems of this kind the nature of the assumptions should be fully explored. The question I wish to raise is rather whether such problems should continue to find a place in our instruction. As the note justly says, they are students' exercises: and that is all they are. Latterly I have been teaching mechanics at Sixth Form level after thirty years spent in other pursuits: and the impression left on me by a renewed acquaintance with school textbooks of mechanics for mathematicians is that their content is governed by the difficulty of constructing problems. Examiners have learnt the trick of constructing problems on certain lines: and boys are trained to solve problems of types chosen for no reason except that these are the problems which examiners have learnt to construct. For example, at the end of the note there is a reference to Newton's Law of Collision. This was an observation of interest at the time when it was made. How accurate the " law " may be I do not know—nor, I suspect, does anybody else: it is of so little importance that a physicist would have to be badly at a loss for occupation before he wasted time in investigating it. But it affords a basis for constructing problems, so the textbooks devote a chapter to dealing with such problems. Except as a test of accuracy of workmanship, a quality which can be tested without bringing in billiard balls, these problems are at their best worthless: and at their worst they require Newton's law to be applied to situations for which it has never been tested, or involve fantastic assumptions such as that one ball hits two other balls simultaneously.

At the Oxford Conference in April, 1957, on Mathematics and Industry there was some talk about getting the physicists and engineers to produce " genuine " problems in mechanics: I would like to know whether any progress is being made to that end.

Yours, etc., J. E. Bullard

53. Ptolemy's Theorem by Area Formulae.

When the area formula "half-product of sides by sine of the included angle" has been taught, a teacher may find it opportune to introduce the analogous formula for the quadrilateral "half-product of diagonals by sine of the included angle". Its proof follows rather easily from a well-known exercise in second year geometry in which a parallelogram is circumscribed to the quadrilateral, having sides parallel to the diagonals. Yet, on account of its limited application at that stage, I had always been at a loss to justify its inclusion as a formula until its unexpected effectiveness and elegance in proving Ptolemy's Theorem appealed to me while reading "Class Room Notes", No. 29; *Mathematical Gazette*, Vol. XLIII, No. 344. The following proof has been adapted to suit the diagram already published.



Let $AE \parallel BD$ and AC intersect BD at O.

$\widehat{CAE} = \alpha_1$, $\widehat{CDE} = \alpha_2$, $\widehat{CBE} = \alpha_3$, $\widehat{COD} = \alpha_4$

Then $\alpha_1 + \alpha_2 = 180°$, and $\alpha_1 = \alpha_3 = \alpha_4$

$\therefore$ $\sin\alpha_1 = \sin\alpha_2 = \sin\alpha_3 = \sin\alpha_4$

also $\triangle BED \equiv \triangle BAD$ ....(i)

$\therefore$ quad. $BEDC$ = quad. $BADC$

and $\triangle BCE + \triangle DCE = ABCD$

$\therefore$ $\frac{1}{2}BC \,.\, BE \sin\alpha_3 + \frac{1}{2}CD \,.\, DE \sin\alpha_2 = \frac{1}{2}AC \,.\, BD \sin\alpha_4$

$\therefore$ $BC \,.\, BE + CD \,.\, DE = AC \,.\, BD$

i.e. $BC \,.\, AD + CD \,.\, AB = AC \,.\, BD$ from (i).

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54. On Note 40 (The Ambiguous Case)

In Fig. 1 quadrilateral $ABCD$ has $AB = DC$ and $\angle BAD = \angle BCD$. To "prove" it a parallelogram draw $BE = BA$. It follows that $BCDE$ is a cyclic trapezium ($\angle AEB = \angle BCD$, $BE = CD$) with BC parallel to ED, and therefore AB is parallel to DC ($\angle BAD = \angle BCD = \text{suppl. } \angle ADC$).

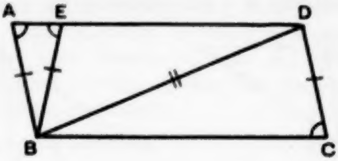

FIG. 1

This proof depends on $AB < BD$. Figs. 2 and 3 show the corresponding structure for $AB > BD$.

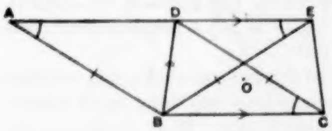

FIG. 2

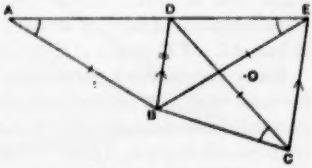

FIG. 3

$BCED$ is a cyclic trapezium ($\angle BED = \angle BCD$, $BE = DC$) with two possibilities (the ambiguous case). Either BC is parallel to DE (Fig. 2), giving $ABCD$ a parallelogram, or BD is parallel to CE (Fig. 3), in which case $ABCD$ is not a parallelogram. O is the centre of circle $BCED$.

The construction in Figs. 4 and 5 is related to the above.

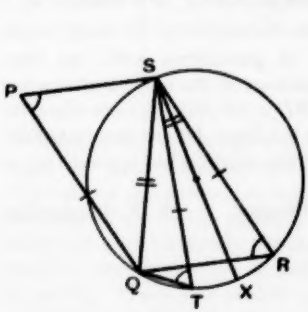

FIG. 4

FIG. 5

$PQRS$ is a parallelogram and SX is the diameter through S of circle QRS. Chord ST is drawn with $\angle XST = \angle XSR$. $PQTS$ is not a parallelogram, but it has $PQ = ST$ and $\angle SPQ = \angle STQ$. This construction requires SR and ST on the same side of SQ, and therefore $PQ > QS$ (the ambiguous case).

The construction in Figs. 6 and 7 convinces first- and second-form pupils of the possibility of a non-parallelogram solution.

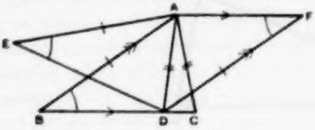

FIG. 6

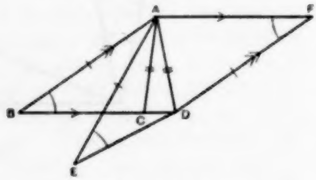

FIG. 7

$\triangle ABC$ has $AC < AB$. Therefore AC and AD ($= AC$) are on the same side of AB. $\triangle ABC$ is rotated to the position of $\triangle AED$. Parallelogram $ABDF$ is drawn. $AEDF$ is not a parallelogram, but it has $AE = FD$ and $\angle AED = \angle AFD$.

Many people deny the existence of the non-parallelogram solution because they can neither visualize nor draw anything other than a parallelogram to meet the required conditions. Most of them begin with parallelogram $ABDF$ (Figs. 6 and 7) and try to draw $\triangle AED$ with $AE = AB$ and $\angle AED = \angle ABD$, which requires $AB > AD$. However, the tendency to draw a parallelogram elongated across the page usually makes $AB < AD$, and consequently a non-parallelogram solution is not found.

Another difficulty arises from the following properties of Figs. 6 and 7. Quadrilateral $AEDF$ (Fig. 6) is convex at A, reduces to $\triangle EDF$, or is concave at A according as $BC \lesseqgtr AB$. Quadrilateral $AEDF$ (Fig. 7) is convex at D, reduces to $\triangle EAF$, or is concave at D according as $BD \lesseqgtr AB$. It follows that the tendency for many people to visualize convex quadrilaterals in association with the term "quadrilateral" obstructs the attainment of the non-parallelogram solution when $BC > AB$ (Fig. 6) and $BD > AB$ (Fig. 7), even when the condition that AC and AD should be less than AB has been satisfied; for, in these cases the non-parallelogram solution has a concavity at one vertex.

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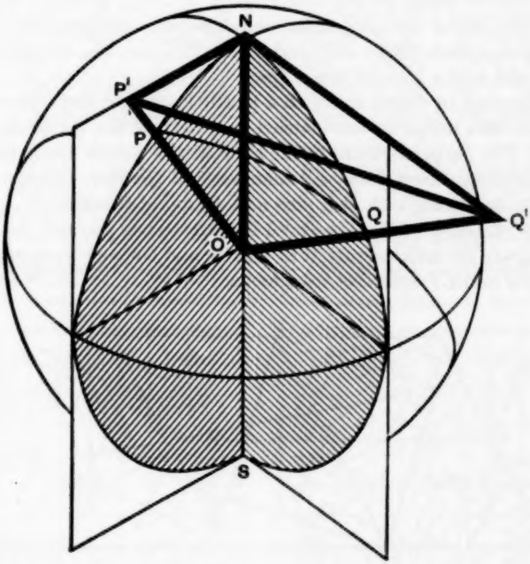
2882. Great Circles on a sphere by drawing

In this age of air travel, most secondary school courses include some study of the geometry of the sphere, with such topics as latitude, longitude and great circles. The distance along a circle of latitude and along a great circle joining two points on the same circle of latitude are often calculated or found by drawing. The general case of the great circle distance between any two points of known latitude and longitude is often omitted, but the diagram below shows that it is scarcely more difficult.



Consider the intersection of the plane of the required great circle with the plane touching the sphere at the north (or south) pole. For ease of description we shall call the latter plane horizontal and the axis of the earth vertical. P and Q are the two points whose latitudes and longitude difference are known. We draw (a) the vertical section through N and P: (b) the vertical section through N and Q. These give us NP' and NQ'. We then draw (c) the horizontal triangle $P'NQ'$, taking $\angle P'NQ'$ as the given difference in longitude. Finally we draw (d) the section through P', Q', P, Q, and O;

this contains the great circle through P and Q. To do this we get OP' and OQ' from (a) and (b) and $P'Q'$ from (c).

The number of minutes in the angle POQ is the great circle distance in nautical miles from P to Q. We can also find from the same figures the most northerly point of the track, or the latitude and longitudes of points dividing the track in any given ratio, and so on. Should PQ cross the equator, we first find the great circle between P and the point antipodal to Q.

R. C. Lyness

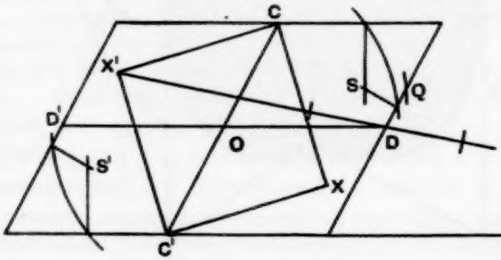
2883. Conjugate diameters and the foci of an ellipse

While investigating the problem of drawing an ellipse to be the parallel projection of a circle, I " discovered " the following property which can be easily proved analytically by using eccentric angles.

If COC', DOD' are conjugate diameters of an ellipse and $CXC'X'$ is a square, then $DX+DX'$ and $DX'-DX$ are the lengths of the major and minor axes of the ellipse.

In drawing an ellipse to touch a parallelogram at the midpoints of its sides, this property can be used to construct the major auxiliary circle. The perpendiculars to the sides at the points where they are cut by the auxiliary circle will then meet at the foci, and the ellipse can now be drawn with two pins and a loop of thread.

Mr. F. Brierley of the Liverpool Instititute has pointed out that if P and Q are on the tangents at C' and D, and $CP = D'Q =$ major axis, then $D'Q$ and CP intersect at a focus.

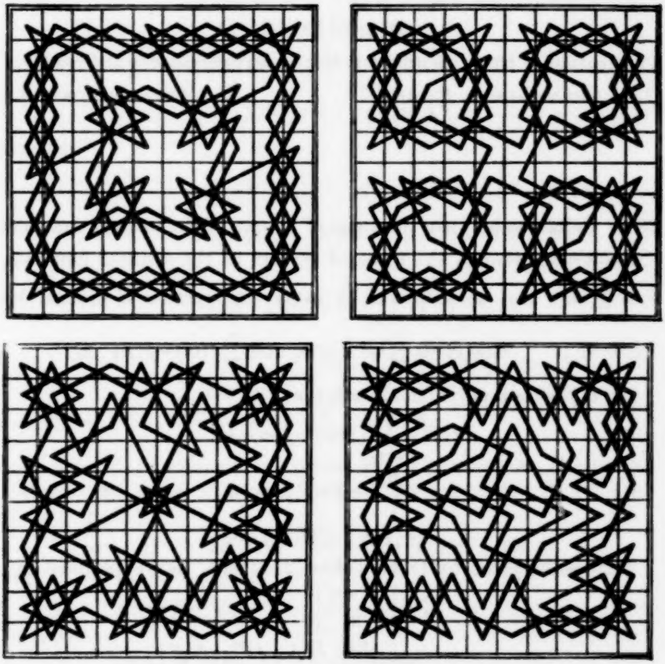


R. C. Lyness

2884. on Note 2592

In February 1956 M. ApSimon posed the question: how many cyclically symmetric knight-tours on a board ten by ten? It appears that the only method of arriving at the exact answer is akin to that which I heard put forward by an American big business man when asked by a somewhat over-subtle English efficiency expert how he would ascertain precisely what percentage of his labour force was productively employed—" Count their durned heads ".

No human computer is likely to have the time or the patience to make the count, but I can now give an estimate, after classifying the tours into types and counting a representative selection. Here are four of the many beautiful results which emerge from this investigation.



The total seems to be in the neighbourhood of 200,000—a surprisingly large number perhaps when one remembers that the number on the only smaller board susceptible to such tours—the six by six—is precisely 5. The next bigger board—fourteen by fourteen—would presumably give hundreds of millions.

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2885. On some diophantine identities

If

$$a^n + b^n + \dots + k^n = a'^n + b'^n + \dots + k'^n$$

for $n = 1, 2, \dots, p$ we write

$$a, b, \dots, k \stackrel{p}{=} a', b', \dots, k'.$$

For any numbers, a, b, c, d such that $a \,.\, b = c \,.\, d$, if $s = a + b + c + d$, we have

$$a, b, s-b, s-a \stackrel{3}{=} c, d, s-d, s-c$$

and

$$(a-b)^2, (s-a)^2, (s-b)^2 \stackrel{2}{=} (c-d)^2, (s-c)^2, (s-d)^2$$

Examples $\quad 2, 6, 9, 13 \stackrel{3}{=} 3, 4, 11, 12$

$$1^2, 18^2, 19^2 \stackrel{2}{=} 7^2, 14^2, 21^2.$$

Verification of these identities is left as an exercise for the reader.

(13*a*) *Gunzenhausen, Bavaria* ALFRED MOESSNER

2886. Relations between the sums of powers of the natural numbers

Relations between the sums of powers of the natural numbers, $S_r = \sum_{m=1}^{n} m^r$, can be obtained by using an operator E defined by $ES_r = S_{r+1}$. This gives $\sum_1^n m^r = E^r S_0$ and $\sum_1^n f(m) = f(E) \,.\, S_0$.

(1) If $f(n)$ is a polynomial such that $f(0) = 0$

$$\begin{aligned} f(n) &= \sum_{m=1}^{n} [f(m) - f(m-1)] \\ &= \sum_1^n f(m) - \sum_1^n f(m-1) \\ &= [f(E) - f(E-1)]S_0. \end{aligned}$$

The results given by D. G. Tahta (2753 Feb. 1958) are given by

$$\begin{aligned} S_1^p &= \left[\frac{n^2+n}{2}\right]^p \\ &= \left[\left(\frac{E^2+E}{2}\right)^p - \left(\frac{E^2-E}{2}\right)^p\right]S_0 \\ &= \frac{1}{2^{p-1}}\left[p\,E^{2p-1} + \binom{p}{3}E^{2p-3} + \dots\right]S_0 \\ &= \frac{1}{2^{p-1}}\left[p\,S_{2p-1} + \binom{p}{3}S_{2p-3} + \dots\right]. \end{aligned}$$

Similarly

$$S_2{}^p = \left[\frac{n(n+1)(2n+1)}{6}\right]^p$$
$$= \frac{E^p}{6^p}\left[(2E^2+3E+1)^p - (2E^2-3E+1)^p\right]S_0.$$

For $p=2, 3$ this gives

$$3S_2{}^2 = 2S_5 + S_3$$
$$12S_2{}^3 = 4S_8 + 7S_6 + S_4.$$

The formula above gives the recurrence relation

$$36S_2{}^{p+2} = 12(2E^3+E)S_2{}^{p+1} - (4E^6 - 5E^4 + E^2)S_2{}^p.$$

Beginning $\quad S_2 = S_2$

$$S_2{}^2 = 3S_5 + S_3$$

we have

$$36S_2{}^3 = 4(2E^3+E)(2S_5+S_3) - (4E^6-5E^4+E^2)S_2$$
$$= 12S_8 + 21S_6 + 3S_4.$$

(2) D. G. Tahta's equation can be written as

$$\frac{(n^2+n)^p}{2} = pS_{2p-1} + \binom{p}{3}S_{2p-3} + \dots$$

Similarly

$$\frac{(2n+1)(n^2+n)^p}{2} = \tfrac{1}{2}[(2E+1)(E^2+E)^p - (2E-1)(E^2-E)^p]S_0$$
$$= \tfrac{1}{2}[2E\{(E^2+E)^p - (E^2-E^p\} + \{(E^2+E)^p + (E^2-E)^p\}]S_0$$
$$= \{2p+1\}S_{2p} + \left\{2\binom{p}{3} + \binom{p}{1}\right\}S_{2p-2} + \left\{2\binom{p}{5} + \binom{p}{3}\right\}S_{2p-4} + \dots;$$

$p=2$ gives

$$\frac{(2n+1)(n^2+n)^2}{2} = 5S_4 + S_2.$$

Both sets of relations are conveniently expressed as recurrence relations, the roots of whose auxiliary equations are E^2+E and E^2-E; thus

$$(n^2+n)^{p+2} = 2E^2(n^2+n)^{p+1} + (E^2-E^4)(n^2+n)^p$$

with

$$(n^2+n) = 2S_1$$
$$(n^2+n)^2 = 4S_3.$$

Therefore

$$(n^2+n)^3 = 2E^2 \,.\, 4S_3 + (E^2-E^4)2S_1$$
$$= 6S_5 + 2S_3$$
$$(n^2+n)^4 = 2E^2 \,.\, (6S_5+2S_3) + (E^2-E^4) \,.\, 4S_3$$
$$= 8S_7 + 8S_5 \text{ and so on.}$$

Also

$$(2n+1)(n^2+n)^{p+2} = 2E^2 \,.\, (2n+1)(n^2+n)^{p+1} + (E^2-E^4) \,.\, (2n+1) \,.\, (n^2+n)^p$$

with $(2n+1)(n^2+n) = 6S_2$

$(2n+1)(n^2+n)^2 = 10S_4 + 2S_2$.

Therefore

$(2n+1)(n^2+n)^3 = 14S_6 + 10S_4$

$(2n+1)(n^2+n)^4 = 18S_8 + 28S_6 + 2S_4$ and so on.

(3) From these results it follows that S_{2r+1} can be expressed as a sum of powers of (n^2+n) and S_{2r} as $2n+1$ multiplied by a series of powers of (n^2+n). It also follows that $(n^2+n)^2$ is a factor of S_{2r+1} for $r \geqslant 1$ and that $(2n+1)(n^2+n)$ is a factor of S_{2r}.

(4) Now n^2+n is $(n+\frac{1}{2})^2 - \frac{1}{4}$, therefore S_{2r} can be expressed as the sum of a series of odd powers of $(n+\frac{1}{2})$ and S_{2r-1} as a sum of even powers.

(5) If $\sum_{m=1}^{n} m^r = f_r(n)$

$\sum_{m=1}^{n-1} m^r = f_r(n-1)$.

But $(n-1)+\frac{1}{2} = -(\frac{1}{2}-n)$.

Therefore

$$f_r(n-1) = (-1)^{r+1} f(-n)$$

But $$f_r(n) - f_r(n-1) = n^r$$

therefore

$$f_r(n) + (-1)^r f(-n) = n^r.$$

When r is even $S_r = \dfrac{n^r}{2} +$ (an odd function of n), and

when r is odd $S_r = \dfrac{n^r}{2} +$ (an even function of n).

(6) The results of paragraphs (4) and (5) can be used to evaluate S_r either as a series of powers of $n+\frac{1}{2}$ or as a series of powers of n.

Setting $S_r = \dfrac{(n+\frac{1}{2})^{r+1}}{r+1} + a_{r-1}(n+\frac{1}{2})^{r-1} + a_{r-3}(n+\frac{1}{2})^{r-3} + \ldots$

a set of simultaneous equations from which the coefficients can be found is given by equating the coefficients of n^{r-2}, n^{r-4}, ... to zero.

Set $S_r = a_{r+1}n^{r+1} + \frac{1}{2}n^r + a_{r-1}n^{r-1} + a_{r-3}n^{r-3} +$

Then, for r odd the odd derivatives are zero for $n = -\frac{1}{2}$ and for r even the expression and its even derivatives are zero for $n = -\frac{1}{2}$.

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2887. Summation of a double series

In a recent paper in the *Journal of the London Mathematical Society* (Vol. 33 (1958), pp. 368–71) Professor Mordell has considered the summation of certain multiple series using a calculus technique. One of his results reduces in the case of a double series to the following:

If $$T(c)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn(m+n+c)}\quad\text{then, for } c>-2,$$

$$T(c)=2\left\{1-\frac{c-1}{1!\,2^3}+\frac{(c-1)(c-2)}{2!\,3^3}-\dots\right\}.$$

It will now be shown that when c is a positive integer $T(c)$ may be evaluated by a repeated application of the standard method for summing series like $\sum\frac{1}{n(n+1)}$, viz. resolution of the general term into partial fractions. Thus,

$$\begin{aligned}
T(c)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m(m+c)}\left(\frac{1}{n}-\frac{1}{n+m+c}\right)\\
&=\sum_{m=1}^{\infty}\frac{1}{m(m+c)}\sum_{n=1}^{m+c}\frac{1}{n}\\
&=\frac{1}{c}\sum_{m=1}^{\infty}\left(\frac{1}{m}-\frac{1}{m+c}\right)\sum_{n=1}^{m+c}\frac{1}{n}\\
&=\frac{1}{c}\sum_{m=1}^{c}\frac{1}{m}\sum_{n=1}^{m+c}\frac{1}{n}+\frac{1}{c}\sum_{m=1}^{\infty}\frac{1}{m+c}\left\{\sum_{n=1}^{m+2c}\frac{1}{n}-\sum_{n=1}^{m+c}\frac{1}{n}\right\}\\
&=\frac{1}{c}\sum_{m=1}^{c}\frac{1}{m}\sum_{n=1}^{m}\frac{1}{n}+\frac{1}{c}\sum_{m=1}^{c}\frac{1}{m}\sum_{n=m+1}^{m+c}\frac{1}{n}+\frac{1}{c}\sum_{m=1}^{\infty}\frac{1}{m+c}\sum_{n=m+c+1}^{m+2c}\frac{1}{n}\\
&=\frac{1}{c}\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}+\frac{1}{c}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m+1}^{m+c}\frac{1}{n}\\
&=\frac{1}{c}\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}+\frac{1}{c}\sum_{n=2}^{c+1}\sum_{m=1}^{\infty}\frac{1}{m(n+m-1)}\\
&=\frac{1}{c}\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}+\frac{1}{c}\sum_{n=2}^{c+1}\sum_{m=1}^{\infty}\frac{1}{n-1}\left(\frac{1}{m}-\frac{1}{m+n-1}\right)\\
&=\frac{1}{c}\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}+\frac{1}{c}\sum_{n=2}^{c+1}\frac{1}{n-1}\sum_{m=1}^{n-1}\frac{1}{m}\\
&=\frac{1}{c}\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}+\frac{1}{c}\sum_{s=1}^{c}\frac{1}{s}\sum_{m=1}^{s}\frac{1}{m}\\
&=\frac{2}{c}\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}.
\end{aligned}$$

To show that this is equivalent to Mordell's result we first note that, for a positive integer n,

$$n-\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)}{3!}-\ldots=1-\left\{\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots\right\}$$
$$=1.$$

Hence $$\frac{1}{n}=1-\frac{n-1}{1!\,2}+\frac{(n-1)(n-2)}{2!\,3}-\ldots.$$

Summing both sides with respect to n and using the result

$$\sum_{r=1}^{n} r(r+1)\ldots(r+s)=n(n+1)\ldots(n+s)(n+s+1)/(s+2), \qquad (*)$$

(see Durell's Advanced Algebra, Vol. I, p. 39), yields

$$\sum_{n=1}^{m}\frac{1}{n}=m-\frac{m(m-1)}{1!\,2^2}+\frac{m(m-1)(m-2)}{2!\,3^2}-\ldots.$$

Hence $$\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}=\sum_{m=1}^{c}\left\{1-\frac{m-1}{1!\,2^2}+\frac{(m-1)(m-2)}{2!\,3^2}-\ldots\right\}$$
$$=c-\frac{c(c-1)}{1!\,2^3}+\frac{c(c-1)(c-2)}{2!\,3^3}-\ldots,$$

by using (*) again.

It should be noted that by writing out the terms of the double series $\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}$ in a triangular array and "completing the rectangle" we can get

$$cT(c)=2\sum_{m=1}^{c}\sum_{n=1}^{m}\frac{1}{mn}=\left\{\sum_{n=1}^{c}\frac{1}{n}\right\}^2+\sum_{n=1}^{c}\frac{1}{n^2}.$$

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2888. A theorem on functionality

The following principle is employed in certain arguments occurring in applied mathematics.

(A) If W and $u_r(r=1,2,\ldots,p)$ are functions of $x_1,\ldots,x_n$, and $dW=\sum_{r=1}^{p}\lambda_r\,du_r$, then W is a function of $u_1,\ldots u_p$, and $\partial W/\partial u_r=\lambda_r$.

For example, in establishing Hamilton's equations of motion, we start with a function H of $q_1,\ldots,q_n,\dot q_1,\ldots,\dot q_n$, and arrive at the relation

$$dH=\sum\dot q_i\,dp_i-\sum\dot p_i\,dq_i;$$

from which it is inferred† that H is a function of $p_1,\ldots,p_n,q_1,\ldots,q_n$ and

$$\frac{\partial H}{\partial p_i}=\dot q_i,\quad\frac{\partial H}{\partial q_i}=-\dot p_i.$$

† Whittaker, *Analytical Dynamics*, §109; or Ramsey, *Dynamics*, II, 235.

We have not found a statement of this principle in the literature; the importance of its applications may therefore make a proof of some interest. It will appear that in general the statement (A) is not true without modification. More precisely, we prove

(B) *If* $W = W(x_1, \dots, x_n)$, $u_r = u_r(x_1, \dots, x_n)$ $(r = 1, \dots, p)$, *where* $x_1, \dots, x_n$ *are independent variables and* W, u_r *are differentiable functions of them, the* u_r *also having continuous first-order partial derivatives, and if also*

$$dW = \sum_{r=1}^{p} \lambda_r \, du_r,$$

then W *is a differentiable function of (some or all of) the* u_r.

Proof. Suppose that, among the p differential forms

$$du_r = \sum_{s=1}^{n} \frac{\partial u_r}{\partial x_s} \, dx_s,$$

m is the maximum number of them which are linearly independent; then $1 < m \leqslant p$ (the case $m = 0$ would be trivial). Choose u-notation so that $du_1, \dots, du_m$ are linearly independent. Then the other $p - m$ forms are linearly dependent on these and so, with new coefficients μ_r, we have

$$dW = \sum_{r=1}^{m} \mu_r \, du_r. \tag{i}$$

The linear independence of $du_1, \dots, du_m$ implies that at least one minor of order m in their $m \times n$ matrix of coefficients $(\partial u_r / \partial x_s)$ is non-zero. Choose x-notation so that its elements are the coefficients of $dx_1, \dots, dx_m$; then

$$\frac{\partial(u_1, u_2, \dots, u_m)}{\partial(x_1, x_2, \dots, x_m)} \neq 0.$$

By the implicit function theorem,‡ the m equations

$$u_r(x_1, \dots, x_n) = u_r \quad (r = 1, \dots, m)$$

determine $x_1, \dots, x_m$ uniquely as differentiable functions of

$$u_1, \dots, u_m, \quad x_{m+1}, \dots, x_n.$$

Substituting for $x_1, \dots, x_m$ in W, we obtain

$$W = W^*(u_1, \dots, u_m, x_{m+1}, \dots, x_n),$$

where W^* (arising from a differentiable function W of differentiable functions) is differentiable in all its variables. Hence

$$dW = \sum_{r=1}^{m} \frac{\partial W^*}{\partial u_r} \, du_r + \sum_{r=m+1}^{n} \frac{\partial W^*}{\partial x_r} \, dx_r. \tag{ii}$$

‡ Goursat, *Cours d'Analyse*, I, 93–97; or Phillips, *Analysis*, 263–5.

$$n - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} - \ldots = 1 - \left\{\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots\right\}$$
$$= 1.$$

Hence $$\frac{1}{n} = 1 - \frac{n-1}{1!\,2} + \frac{(n-1)(n-2)}{2!\,3} - \ldots .$$

Summing both sides with respect to n and using the result

$$\sum_{r=1}^{n} r(r+1)\ldots(r+s) = n(n+1)\ldots(n+s)(n+s+1)/(s+2), \qquad (*)$$

(see Durell's Advanced Algebra, Vol. I, p. 39), yields

$$\sum_{n=1}^{m} \frac{1}{n} = m - \frac{m(m-1)}{1!\,2^2} + \frac{m(m-1)(m-2)}{2!\,3^2} - \ldots .$$

Hence $$\sum_{m=1}^{c} \sum_{n=1}^{m} \frac{1}{mn} = \sum_{m=1}^{c} \left\{1 - \frac{m-1}{1!\,2^2} + \frac{(m-1)(m-2)}{2!\,3^2} - \ldots\right\}$$
$$= c - \frac{c(c-1)}{1!\,2^3} + \frac{c(c-1)(c-2)}{2!\,3^3} - \ldots ,$$

by using (*) again.

It should be noted that by writing out the terms of the double series $\sum_{m=1}^{c} \sum_{n=1}^{m} \frac{1}{mn}$ in a triangular array and "completing the rectangle" we can get

$$cT(c) = 2 \sum_{m=1}^{c} \sum_{n=1}^{m} \frac{1}{mn} = \left\{\sum_{n=1}^{c} \frac{1}{n}\right\}^2 + \sum_{n=1}^{c} \frac{1}{n^2} .$$

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2888. A theorem on functionality

The following principle is employed in certain arguments occurring in applied mathematics.

(A) If W and $u_r (r = 1, 2, \ldots, p)$ are functions of $x_1, \ldots, x_n$, and $dW = \sum_{r=1}^{p} \lambda_r\, du_r$, then W is a function of $u_1, \ldots u_p$, and $\partial W/\partial u_r = \lambda_r$.

For example, in establishing Hamilton's equations of motion, we start with a function H of $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n$, and arrive at the relation

$$dH = \sum \dot{q}_i\, dp_i - \sum \dot{p}_i\, dq_i ;$$

from which it is inferred† that H is a function of $p_1, \ldots, p_n, q_1, \ldots, q_n$ and

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i .$$

† Whittaker, *Analytical Dynamics*, §109; or Ramsey, *Dynamics*, II, 235.

We have not found a statement of this principle in the literature; the importance of its applications may therefore make a proof of some interest. It will appear that in general the statement (A) is not true without modification. More precisely, we prove

(B) *If* $W = W(x_1, \dots, x_n)$, $u_r = u_r(x_1, \dots, x_n)$ $(r = 1, \dots, p)$, *where* $x_1, \dots, x_n$ *are independent variables and* W, u_r *are differentiable functions of them, the* u_r *also having continuous first-order partial derivatives, and if also*

$$dW = \sum_{r=1}^{p} \lambda_r \, du_r,$$

then W *is a differentiable function of (some or all of) the* u_r.

Proof. Suppose that, among the p differential forms

$$du_r = \sum_{s=1}^{n} \frac{\partial u_r}{\partial x_s} \, dx_s,$$

m is the maximum number of them which are linearly independent; then $1 < m \leq p$ (the case $m = 0$ would be trivial). Choose u-notation so that $du_1, \dots, du_m$ are linearly independent. Then the other $p - m$ forms are linearly dependent on these and so, with new coefficients μ_r, we have

$$dW = \sum_{r=1}^{m} \mu_r \, du_r. \tag{i}$$

The linear independence of $du_1, \dots, du_m$ implies that at least one minor of order m in their $m \times n$ matrix of coefficients $(\partial u_r / \partial x_s)$ is non-zero. Choose x-notation so that its elements are the coefficients of $dx_1, \dots, dx_m$; then

$$\frac{\partial(u_1, u_2, \dots, u_m)}{\partial(x_1, x_2, \dots, x_m)} \neq 0.$$

By the implicit function theorem,‡ the m equations

$$u_r(x_1, \dots, x_n) = u_r \quad (r = 1, \dots, m)$$

determine $x_1, \dots, x_m$ uniquely as differentiable functions of

$$u_1, \dots, u_m, \quad x_{m+1}, \dots, x_n.$$

Substituting for $x_1, \dots, x_m$ in W, we obtain

$$W = W^*(u_1, \dots, u_m, x_{m+1}, \dots, x_n),$$

where W^* (arising from a differentiable function W of differentiable functions) is differentiable in all its variables. Hence

$$dW = \sum_{r=1}^{m} \frac{\partial W^*}{\partial u_r} \, du_r + \sum_{r=m+1}^{n} \frac{\partial W^*}{\partial x_r} \, dx_r. \tag{ii}$$

‡ Goursat, *Cours d'Analyse*, I, 93–97; or Phillips, *Analysis*, 263–5.

Comparing (i) and (ii), we have an identity

$$\sum_{r=1}^{m} \mu_r\, du_r = \sum_{r=1}^{m} \frac{\partial W^*}{\partial u_r} du_r + \sum_{r=m+1}^{n} \frac{\partial W^*}{\partial x_r} dx_r$$

in which $du_1, \ldots, du_m, dx_{m+1}, \ldots, dx_n$ are arbitrary. Equating differential coefficients,

$$\mu_r = \frac{\partial W^*}{\partial u_r} \quad (r = 1, \ldots, m)$$

and

$$0 = \frac{\partial W^*}{\partial x_r} \quad (r = m+1, \ldots, n).$$

The last set of equations shows that W^* is independent of $x_{m+1}, \ldots, x_n$. Hence W is a function W^* of $u_1, \ldots, u_m$; and further, μ_r is identified as $\partial W/\partial u_r$ $(r = 1, \ldots, m)$.

In the dynamical example above, $W \equiv H = \sum p_i \dot{q}_i - L$, where $u_i \equiv p_i = \partial L/\partial \dot{q}_i$ and $L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$. In a natural dynamical system the part of L dependent on $\dot{q}_1, \ldots, \dot{q}_n$ is a positive definite quadratic form in these variables, so that

$$\frac{\partial(p_1, \ldots, p_n)}{\partial(\dot{q}_1, \ldots, \dot{q}_n)} > 0$$

and the result (B) certainly applies to give $H = H(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$. Consideration of an abstract system could lead however to a function L of quite different type; and for the usual deductions to be possible, a determinantal condition $|\, \partial^2 L/\partial \dot{q}_i\, \partial \dot{q}_j \,| \neq 0$ would be essential.

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2889. A new vector transformation

In a paper Durand* obtains by tensor methods the vector relation

$$\int_c \mathbf{ds} \wedge \mathbf{f} = \iint_s \{(\mathbf{n} \cdot \nabla)\mathbf{f} + \mathbf{n} \wedge \operatorname{curl} \mathbf{f} - \mathbf{n} \operatorname{div} \mathbf{f}\}\, dS,$$

with the usual notation. As far as the writer is aware this transformation, which has some useful applications in Applied Mathematics, occurs in texts only in tensor or dyadic form. The purpose of this note is to give a proof using only vector methods.

With rectangular cartesian axes we can write

$$\begin{aligned} \mathbf{ds} \wedge \mathbf{f} &= (\mathbf{ds} \wedge \mathbf{f} \cdot \mathbf{i})\mathbf{i} + (\mathbf{ds} \wedge \mathbf{f} \cdot \mathbf{j})\mathbf{j} + (\mathbf{ds} \wedge \mathbf{f} \cdot \mathbf{k})\mathbf{k} \\ &= (\mathbf{f}' \cdot \mathbf{ds})\mathbf{i} + (\mathbf{f}'' \cdot \mathbf{ds})\mathbf{j} + (\mathbf{f}''' \cdot \mathbf{ds})\mathbf{k} \end{aligned}$$

* E. Durand, "A new vector transformation of a curvilinear integral into a surface integral and its application to magneto-statics," *Comptes Rendus*, 241 (1955) 594–6.

where

$$\mathbf{f}' = \mathbf{f}\wedge\mathbf{i}, \quad \mathbf{f}'' = \mathbf{f}\wedge\mathbf{j} \quad \text{and} \quad \mathbf{f}''' = \mathbf{f}\wedge\mathbf{k}. \tag{1}$$

Thus

$$\int_c \mathbf{ds}\wedge\mathbf{f} = \int_c (\mathbf{f}'\cdot\mathbf{ds})\mathbf{i} + \int_c (\mathbf{f}''\cdot\mathbf{ds})\mathbf{j} + \int_c (\mathbf{f}'''\cdot\mathbf{ds})\mathbf{k}$$

$$= \int_s (\operatorname{curl}\mathbf{f}'\cdot\mathbf{n})\mathbf{i}\,dS + \text{two similar expressions}. \tag{2}$$

Substituting for $\mathbf{f}'$ from (1) we find that

$$\int_s (\operatorname{curl}\mathbf{f}'\cdot\mathbf{n})\mathbf{i}\,dS = \int_s \left\{\left(\mathbf{n}\cdot\frac{\partial\mathbf{f}}{\partial x}\right)\mathbf{i} - ((\mathbf{n}\cdot\mathbf{i})\operatorname{div}\mathbf{f})\mathbf{i}\right) dS.$$

On treating the remaining members of (2) in a similar way and adding,

$$\int_c \mathbf{ds}\wedge\mathbf{f} = \int_s \left\{\left(\mathbf{n}\cdot\frac{\partial\mathbf{f}}{\partial x}\right)\mathbf{i} + \left(\mathbf{n}\cdot\frac{\partial\mathbf{f}}{\partial y}\right)\mathbf{j} + \left(\mathbf{n}\cdot\frac{\partial\mathbf{f}}{\partial z}\right)\mathbf{k} - \mathbf{n}\operatorname{div}\mathbf{f}\right\} dS.$$

Let $\mathbf{n} = (n_1, n_2, n_3)$ and $\mathbf{f} = (f_1, f_2, f_3)$; then

$$\left(\mathbf{n}\cdot\frac{\partial\mathbf{f}}{\partial x}\right)\mathbf{i} = \left[\left(n_1\frac{\partial f_1}{\partial x} + n_2\frac{\partial f_2}{\partial x} + n_3\frac{\partial f_3}{\partial x}\right)\right.$$

$$\left. + \left(n_2\frac{\partial f_1}{\partial y} + n_3\frac{\partial f_1}{\partial z}\right) - \left(n_2\frac{\partial f_1}{\partial y} + n_3\frac{\partial f_1}{\partial z}\right)\right]\mathbf{i}$$

$$= (\mathbf{n}\cdot\nabla)f_1\mathbf{i} + \left\{n_2\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - n_3\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right\}\mathbf{i}$$

$$= (\mathbf{n}\cdot\nabla)f_1\mathbf{i} + (x\text{-component of } \mathbf{n}\wedge\operatorname{curl}\mathbf{f})\mathbf{i}.$$

Similar expressions are obtained for

$$\left(\mathbf{n}\cdot\frac{\partial\mathbf{f}}{\partial y}\right)\mathbf{j} \quad \text{and} \quad \left(\mathbf{n}\cdot\frac{\partial\mathbf{f}}{\partial z}\right)\mathbf{k},$$

and on adding we have

$$\int_c \mathbf{ds}\wedge\mathbf{f} = \iint_s \{(n\cdot\nabla)\mathbf{f} + \mathbf{n}\wedge\operatorname{curl}\mathbf{f} - \mathbf{n}\operatorname{div}\mathbf{f}\}\,dS.$$

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2890. A propos de la note 2862 (*M.G.*, XLIII, p. 198).

La remarque indiquée dans cette note (No. 2862) n'est pas nouvelle. Voir, par example:

H. Lass, *Elements of pure and applied mathematics*, New York, 1957, p. 45.

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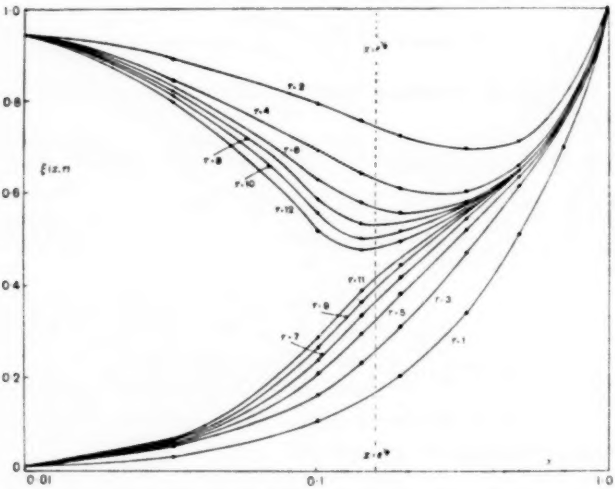
2891. Iterated Exponentials

We have made a brief investigation into some of the elementary properties of the function

$$\text{(1)} \qquad \xi(x, r) = x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}} \Big\} \; r \text{ times}$$

where $x \geqslant 0$ and is real, and r is an integer $\geqslant 0$.



This work was done initially in ignorance of that carried out by R. L. Goodstein and P. Goodstein (Notes 2718, 2803 and 2804). We are now greatly indebted to Professor Goodstein for the rigorous proofs of the properties we give below; essentially this is his work and his name (not ours) should appear beneath it.

The properties we shall prove are:

1 $\quad \lim_{x \to 0} \xi(x, r) = 0, \quad r$ odd,

$\qquad\qquad = 1, \quad r$ even.

2 $\quad$ For $n \geqslant r$, $[D^r \xi(x, n)]_{x=1} = [D^r y]_{x=1}$ where $y = x^y$.

The first 6 values $[D^r y]$ at $x = 1$ are 1, 2, 9, 56, 480, 4914. $\xi(x, r)$ is shown plotted in the accompanying figure for several values of r, and for x in the range $0{\cdot}01 \leqslant x \leqslant 1$.

1. The proof is by induction.

$$\xi(x, 1) = x \to 0 \quad \text{as } x \to 0$$

$$x^x = e^{x \log x} \to e^0 = 1 \quad \text{as } x \to 0, \quad \text{i.e.} \quad \xi(x, 2) \to 1 \quad \text{as } x \to 0.$$

Suppose $\quad \xi(x, 2r-1)\to 0,\ \xi(x, 2r)\to 1$ for some r.

Then $\quad \xi(x, 2r+1) = e^{\xi(x,2r)\log x} \to e^{-\infty} = 0 \quad$ as $x\to 0$.

Since $\xi(x, 2r)\to 1$, for small enough x, $\xi(x, 2r) > \frac{1}{2}$ and so $0 < x < \delta$

$$|\log \xi(x, 2r+2)| = x^{\xi(x,2r)}|\log x| < x^{\frac{1}{2}}|\log x| \to 0$$

as $x\to 0$. Therefore $\xi(x, 2r+2)\to 1$, whence the result is proved by induction.

2. Let y be a function which is differentiable infinitely often and write $D^r x^y = \phi_r(x, y, y', y'' \dots y^r)$ where $y^s = D^s y$

Because $D^{r+1} x^y = D^r(Dx^y) = D^r\left\{\left(y'\log x + \dfrac{y}{x}\right)x^y\right\}$ it follows that the coefficient of y^{r+1} in $D^{r+1}x^y$ is $(\log x)x^y$ which vanishes when $x=1$.

With this lemma we prove the result required by induction.

By direct differentiation we can shew easily,

$$[D\xi(x,1)]_{x=1} = [Dy]_{x=1}$$

$$[D\xi(x,2)]_{x=1} = [Dy]_{x=1},\ [D^2\xi(x,2)]_{x=1} = [D^2y]_{x=1}$$

We assume for some n

$$[D^r\xi(x,n)]_{x=1} = [D^ry]_{x=1}, \quad 1\leqslant r\leqslant n$$

Then $\quad D^{r+1}\xi(x, n+1) = D^{r+1}(x^{\xi(x,n)})$

$$= \phi_{r+1}(x, \xi, \xi', \dots, \xi^{r+1}), \quad \text{where } \xi = \xi(x, n)$$

and $\quad D^{r+1}y = D^{r+1}x^y = \phi_{r+1}(x, y, y', \dots y^{r+1})$.

But at $x=1$, $\xi = y$, $\xi' = y'$, …, $\xi^r = y^r$, and the terms in ξ^{r+1}, y^{r+1} vanish, so that

$$[D^{r+1}\xi(x, n+1)]_{x=1} = [D^{r+1}y]_{x=1}$$

and the result is proved.

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2892. A remark about l'Hospital's rule

In the elementary theory of limits, of functions of a real variable, much that is true for real-valued functions remains true when the values are allowed to be complex. Where infinite limits are involved however, the two cases differ in a way that can be illustrated rather strikingly by considering l'Hospital's rule.

A general form of this rule can be stated as follows. *Let f and g be functions (of a real variable x) such that, as $x\to 0$,* (i) *$f(x)\to c$ and $g(x)\to c$, where c is either 0 or ∞,* (ii) *$f'(x)\to a$ and $g'(x)\to b$, where a and b are not both 0 or both ∞. Then $f(x)/g(x)\to a/b$.* When f and g are

real-valued functions, the validity of this can be established by a simple application of the mean-value theorem. The proof can easily be adapted to the case of complex-valued functions if it is known that the real and imaginary parts of $f'(x)$ and $g'(x)$ tend separately to limits as $x \to 0$; this will be so *provided that a and b are both finite.* Without this proviso, the rule may fail in the complex case. Suppose, for example, that, when $x \neq 0$, $f(x) = x$ and $g(x) = xe^{i/x}$: then $c = 0$, $a = 1$, and $b = \infty$ (so that $a/b = 0$); but $f(x)/g(x)$ does not tend to any limit as $x \to 0$.

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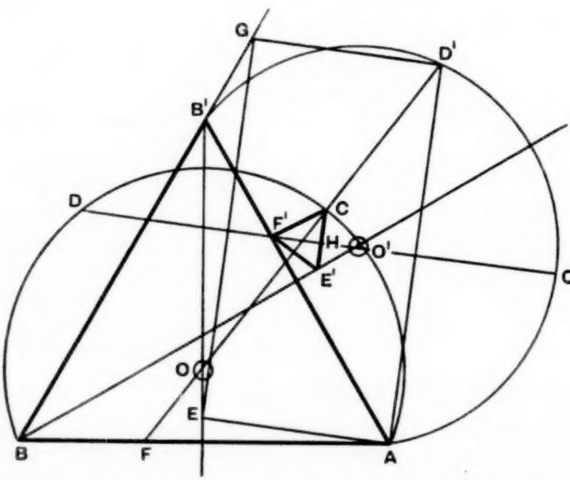
2893. On two equal arcs on the sides of an equilateral triangle

Equal arcs $ACDB$, $AC'D'B'$ are described on the sides AB, AB' of an equilateral triangle ABB', with centres O, O' respectively. Arc AB is trisected at C, D and arc AB' is trisected at C', D'. CD' meets AB at F. $C'D$ meets AB' at F', $C'F$ meets OB' at E and E' is chosen on BO so that $BD = DE'$, G is chosen on BB' so that $C'G = C'B'$. *Then $AEGD'$ is a rectangle and $CE'F'$ is an equilateral triangle.*



Proof. We start with the observation that CD' passes through O, and $C'D$ through O'. For if arc $AC'B'$ meets OB' at H', then, since angle $AB'H' = 30°$, $H'A = O'A = OA$, and therefore H' coincides with O, i.e. arc $AC'B'$ passes through O, and similarly arc ACB passes through O'. Hence if OC meets arc AB' at D'', then AD'' subtends at O' an angle equal to twice angle AOC, and so arc AD'' equals

twice arc AC proving that D'' coincides with D'. Similarly $C'D$ passes through O'.

Next we prove that $AF = AF'$; rotate arc AB' round A through $60°$ so that C' comes into coincidence with C and O' into coincidence with O, and therefore F' coincides with F, showing that $AF = AF'$. Let arc AB' subtend an angle 3α at O, so that each of AC', $C'D'$, $D'B'$ subtends an angle α at O, and 2α at O'. Then angle $C'AB' = 2\alpha$, angle $AC'O' =$ half angle $AC'D' = 90° - \alpha$, and therefore angle $AF'C' = 180° - (90° - \alpha) - 2\alpha = 90° - \alpha =$ angle $AC'F'$ proving that $AF' = AC' = AC$. It is now easy to show that triangle $CE'F'$ is equilateral. Let O lie outside ABB'. We have angle $BDE' = 180° - 2(30° - 2\alpha) = 120° + 4\alpha$, angle $BDC = 180° - 2\alpha$ and therefore angle $CDE' = 60° - 2\alpha$. But $CAF' = 60° - 2\alpha$ and $AC = AF' = DC = DE'$ so that triangles DCE', ACF' are congruent, making $CE' = CF'$. Moreover angle $DCE' = 30° + \alpha$ and so angle $E'CF' = 60°$, which completes the proof that $E'CF'$ is equilateral.

Choose G' on the perpendicular to AD' through D' (on the same side of AD' as B') so that $C'G' = C'B'$; then from triangle $C'D'G'$

$$(\sin C'G'D')/CD' = \sin(90° + \alpha)/C'G'$$

whence $\sin C'G'D' = \sin\alpha\cos\alpha/\sin 2\alpha = 1/2$ showing that

$$\angle C'G'D' = 30°.$$

Hence angle $C'G'B' = 60° + \alpha =$ angle $C'B'G'$ showing that $OB'G' = 30°$ and therefore G' is the same point as G.

Choose K on OB' so that AK is perpendicular to AD'. From triangle $AB'K$, $AK = (AB' \sin 30°)/\sin(60° + \alpha)$ and from triangle $C'D'G$, $D'G = \sin(60° - \alpha) . C'D'/\sin 30°$ so that

$$D'G/AK = 4(\sin^2 60° - \sin^2\alpha)\sin\alpha/\sin 3\alpha = 1.$$

Thus $AKGD'$ is a rectangle. It remains to show that K is the same point as E.

Let M be the foot of the perpendicular from A to OB', and let $C'K$ meet AM at L. The triangles AKC', $D'GC'$ are congruent and so angle $AKC' = 30°$, and therefore angle $ALC' = 30° + (30° - \alpha) =$ angle $AC'L$, whence $AL = AC'$. Hence $LM/OM = (\sin 3\alpha - 2\sin\alpha)\sec 3\alpha = \tan\alpha$ and therefore angle $LOM = \alpha$ proving that L is the same point as F and hence K is the same point as E.

I am indebted to the Editor for this proof.

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2894. Geodesics on a sphere

This treatment of the problem of finding the shortest arc on the surface of a sphere joining two given points on it avoids the use of variational methods and does not require any knowledge of the more advanced techniques of differential geometry.

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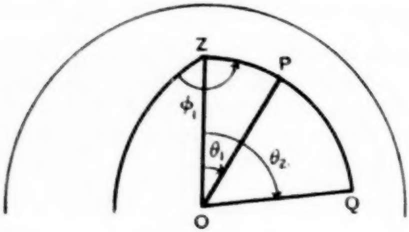

Let P and Q be two points on the surface of a sphere of radius a and centre O. Choose any radius OZ of the sphere in the plane OPQ and denote the angles ZOP, ZOQ by θ_1, θ_2. Suppose this plane makes azimuth angle ϕ_1 with some other chosen reference plane. Then the spherical polar coordinates of P and Q are (a, θ_1, ϕ_1), (a, θ_2, ϕ_1).

Without loss of generality we suppose $\theta_2 > \theta_1$. Then the length of any arc joining P to Q is given by

$$l = a\int_{\theta_1}^{\theta_2} [(d\theta)^2 + (\sin\theta\, d\phi)^2]^{\frac{1}{2}} = a\int_{\theta_1}^{\theta_2}\left[1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta.$$

Since the integrand is ≥ 1, the length l is least if the integrand can be chosen to be unity all along the arc considered. This requires $\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2 = 0$ along the admissible arc, which can only be satisfied by choosing $\left(\frac{d\phi}{d\theta}\right) = 0$ along the length of the arc, since $\sin\theta \neq 0$ for $\theta_1 \leq \theta \leq \theta_2$. This is satisfied by $\phi =$ const along the entire arc and hence $\phi = \phi_1$, the common value of ϕ at P and Q. Hence each point of the minimum arc PQ has the same azimuth angle ϕ_1 and so the arc is the smaller section of the great circle through P and Q.

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2895. Integral-sided triangles.

Problems in which the answers " come out nicely ", although frowned upon in some quarters, have a certain value in practical teaching, as they give the beginner confidence in applying a new formula or method, where heavy calculations prevent him seeing the wood for the trees.

1. *Pythagorean triangles.* The general formulae are very well-known, but are mentioned for the sake of completeness: $a^2 + b^2 = c^2$ if $a = l^2 - m^2$, $b = 2lm$, $c = l^2 + m^2$ (where to avoid solutions reducible to simpler ones l, m are co-prime and not both odd).

2. *Heronian triangles.* These have integral area, and hence rational altitudes and radii of in-, ex- and circumcircles. Numerous applications, *e.g.* to the s-formula for the area and to easily-checked elementary constructions, will suggest themselves. The expressions $l+m$, $m+n$, $n+l$ where $l+m+n=lmn$ give rational values for the sides, but it is much easier to make Heronian triangles by fitting together Pythagorean triangles with equal sides (adjacent to the right angles). Such triangles, reduced if necessary, will always have one integral altitude.

3. *Apollonian triangles.* These are triangles with integral sides and (at least) one integral median. A convenient formula for finding these is: $a^2+b^2=2(u^2+v^2)$ if $a=l+m+n-p$, $b=l+m-n+p$, $u=l-m$, $b=n+p$, provided $lm=np$.

4. $60°$ *and* $120°$ *triangles.* Integral sided examples of these are useful for making up exercises on the cosine rule. They can be found from the formulae:

If
$$a = 3l^2 - 2lm - m^2$$
$$a' = 4lm$$
$$b = 3l^2 + 2lm - m^2 \quad (=a+a')$$
$$c = 3l^2 + m^2$$
then:
$$\begin{aligned} c^2 &= a^2 + b^2 - ab \\ &= a'^2 + b^2 - a'b \\ &= a^2 + a'^2 + aa' \end{aligned}$$
so that in $\triangle$'s a, b, c; a', b, c; $C = 60°$, and in $\triangle a, a', c$; $C = 120°$.

A. R. PARGETER

2896. Supplement to Note 2895.

$60°$ and $120°$ triangles can also be found as follows:

If
$$c^2 = a^2 \pm ab + b^2 = (a \mp \omega b)(a \mp \omega^2 b).$$
then
$$\pm(a+\omega b) = (u+\omega v)^2 = u^2 + 2uv\omega - (1+\omega)v^2.$$
We have, therefore
$$a = |u^2 - v^2|,\ b = |2uv - v^2|$$
and
$$c = u^2 - uv + v^2.$$
These formulae give $60°$ or $120°$ triangles according to the relative magnitude of u and v. Since
$$(u^2 - uv + v^2)^2 = (u^2 - v^2)^2 + (2uv - v^2)^2 - (u^2 - v^2)(2uv - v^2),$$
the angle will be $60°$ if, for $v>0$, either $u>v$ or $\frac{1}{2}v>u>-v$, and $120°$ otherwise.

H. M. C.

2897. On the equation ax − by = 1

The object of this note is to describe a method of solving the diophantine equation

$$ax - by = 1$$

which is independent of Euclid's algorithm, and continued fractions, and could be used by young children. I start with an application of the method to the numerical example:

$$5x - 11y = 1;$$

we consider the permutation

$$t \to t + 5 \pmod{11}$$

which is conveniently expressed by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

(in which we note that " 5 " is entered beneath " 11 " and the rest is then filled in mechanically). From this permutation we select the cycle which begins with 5 in the first row and ends with 1 in the second, viz.

$$5,\ 10,\ 4,\ 9,\ 3,\ 8,\ 2,\ 7,\ 1.$$

This cycle contains 9 terms, and $x = 9$ is a solution of the equation $5x - 11y = 1$; the corresponding value of y is given by the number of times a term in the cycle is followed by a smaller number, which in this case is 4 (of course $y = (5 \times 9 - 1)/11$ simply). To find a solution of

$$5x - 11y = 3$$

we consider the same permutation and select the cycle which ends with 3, namely

$$5,\ 10,\ 4,\ 9,\ 3$$

which contains 5 terms, and 2 decreases, so that

$$x = 5,\ y = 2 \text{ is a solution.}$$

As another example I consider the equation

$$7x - 5y = 1$$

in which the coefficient of x is now the larger of the two coefficients; this time we use the permutation

$$a \to a - 5 \pmod 7$$

from which we select the cycle

$$7,\ 2,\ 4,\ 6,\ 1;$$

this cycle contains *five* terms, one more than the value of y, and the terms increase *twice*, once less than the value of x. Of course this second case can always be brought under the first, by writing it in the form

$$2x-5(y-x)=1;$$

the cycle now is 2, 4, 1 giving x the value 3, and $y-x$ the value 1, so that $y=4$.

We consider now the general equation

$$ax-by=c$$

with each two of a, b, c relatively prime and $a>b>c$. The permutation

$$t \to t+a \pmod{b}$$

gives rise to the cycle

$$a, 2a, 3a, \dots, c$$

provided that there is a natural number k such that $ka \equiv c \pmod{b}$; the b numbers $a, 2a, 3a, \dots, (b-1)a, ba$ on division by b leave the remainders $0, 1, 2, \dots, b-1$ in some order, for if ra and sa, $1 \leqslant r < s \leqslant b$, left the same remainder on division by b then $(s-r)a$ would be divisible by b, which is impossible, since a and b have no common factor and $s-r<b$, so that $s-r$ is not divisible by b. Thus there is a value of k between 1 and $b-1$ (inclusive) such that $ka=c$. This value of k gives the number of terms in the cycle

$$a, 2a, \dots, ka \equiv c \pmod{b},$$

and since by hypothesis $ka-c$ is divisible by b, there is a number l such that

$$ka-c=lb$$

i.e.

$$ka-lb=c$$

showing that $x=k$, $y=l$ is a solution of the equation

$$ax-by=c.$$

It remains to show that l is the number of times a term in the cycle

$$a, 2a, \dots, c \pmod{b}$$

is followed by a smaller number. Now a term t in the cycle is followed by the larger number $t+a$ if $t+a \leqslant b$ and is followed by a smaller number if $t+a>b$; each term in the cycle has the form $pa-qb$ and the last term is

$$ka-lb=c$$

so that l counts the number of times b has been subtracted, i.e. the number of times a term has decreased. A similar argument applies when $a>b$. To ensure that this treatment of the equation $ax-by=c$ really is independent of Euclid's algorithm, we must check that no step in the proof contains a concealed application of the algorithm. In fact the critical step in the proof which consists in showing that a prime number which divides a product da and does not divide a, necessarily divides d, is generally proved by an application of the algorithm to establish the uniqueness of prime factorisation, but

there are proofs (of the theorem that if p divides da and p does not divide a then p divides d) which do not use the algorithm; a very simple one is given by Davenport in his splendid *Introduction to the Higher Arithmetic* (in the Hutchinson series).

The result may also be obtained very simply as follows. If there is a prime number which divides some product uv without dividing either of the factors u, v, let p be the smallest such prime and let ab be the smallest product which p divides without dividing either of the factors. Necessarily a and b are less than p for if $a>p$ then $(a-p)b$ is a smaller product divisible by p and such that neither of the factors $a-p$, b is divisible by p. Since ab is divisible by p, there is q such that

$$ab=pq$$

Let f be a prime factor of a, then $f<p$ and f divides pq so that f divides q (for p is prime); let a', q' be the quotients when a, q are divided by f, so that

$$fa'b=fpq'$$

and therefore

$$a'b=pq';$$

thus $a'b$ is a smaller product divisible by p and such that neither of the factors a', b is divisible by p. This contradiction proves that there is no prime which divides a product without dividing one of the factors.

By comparison with the standard method of solution of a diophantine equation, finding a cycle of the appropriate permutation may be absurdly long and tiresome. For instance, in the case of the equation

$$79x-101y=1$$

the cycle we require contains 78 terms, whereas, using the algorithm, we obtain the sequence 101, 79, 22, 13, 9, 4, 1, 3, 7, 10, 17, 61, 78 of only 13 terms (and do not in fact need to count the number of terms).

The existence of the cycle ending in unity, furnishes a simpler proof of the existence of a solution, but it must be observed that the existence of a cycle in turn rests upon the existence of a solution of the congruence $az\equiv 1 \pmod{b}$ and the existence of a solution of this congruence provides a direct proof of the existence of the solution $x=z$ of the equation

$$ax-by=1$$

In solving diophantine equations on a digital computer, however, using a permutation leads to a simpler programme than using Euclid's algorithm.

R. L. GOODSTEIN

2898. On Note 2454

Recently rereading my Note 2454 I thought of the following immediate derivation of the addition theorems for the circular functions from the differential equations

$$\frac{d^2s}{dx^2}+s=0,\quad \frac{d^2c}{dx^2}+c=0$$

with the initial conditions $s(0)=c'(0)=0$, $c(0)=s'(0)=1$. The derivative of the function

$$c(x)s'(a-x)+c'(x)s(a-x)$$

is zero, and so equating the function to its value at $x=0$, we find

$$c(x)s'(a-x)+c'(x)s(a-x)=s'(a); \tag{1}$$

taking $x=a$ in this equation gives

$$s'(a)=c(a)$$

and differentiating this last equation yields $c'(a)=-s(a)$, whence substituting these values in (1) and giving a the value $x+y$ we find

$$c(x+y)=c(x)c(y)-s(x)s(y),$$

and after another differentiation

$$s(x+y)=s(x)c(y)+c(x)s(y).$$

The same method applies of course to the hyperbolic functions.

R. L. Goodstein

2899. An interesting configuration.

The following configuration could stimulate the future geometer at the sixth form if drawn for pleasure, or dishearten him if set as an examination question. The proof can be carried out by the use of the parametric representation of the rectangular hyperbola.

An expression such as $(ABCD, P)$ denotes a circle through $ABCD$ with centre at P, A' denotes the point symmetric to A with respect to the origin.

A circle with centre at P and radius r meets the rectangular hyperbola $xy=1$ at $ABCD$. Then we have the following eight pairs of circles:

$$(ABCD, P)(A'B'C'D', P');\quad (ABC'D', Q)(A'B'CD, Q');$$
$$(AB'CD', R)(A'BC'D, R');\quad (AB'C'D, S)(A'BCD', S').$$
$$(PQRS, A)(P'Q'R'S', A');\quad (PQR'S', B)(P'Q'RS, B');$$
$$(PQ'RS', C)(P'QR'S, C');\quad (PQ'R'S, D)(P'QRS', D').$$

$ABCDA'B'C'D'$ are on the rectangular hyperbola $xy=1$.

$$PQRSP'Q'R'S'$$

are on a rectangular hyperbola whose asymptotes bisect those of the rectangular hyperbola $xy=1$. All the circles have the same radius r.

Each point is the centre of a circle and through each point pass 4 circles. Each point is the orthocentre of 4 triangles: e.g. A' is the orthocentre of BCD, $BC'D'$, $B'CD'$, $B'C'D$. There are two sets of 24 triangles each.

Box 262, Kampala, Uganda I. M. KHABAZO

2900. A Classical Problem.

Many mathematicians have attempted to obtain a general solution of the problem of moving a knight over a square board—visiting each square once and only once and ending on a square at a knight's move from the starting point. I have obtained three distinct general solutions; but considerations of space make it quite impossible to outline the solutions here. However the beauty and simplicity of the "quarter-symmetric" solution for a $(4n+2)^2$ board is illustrated in the following diagrams. The first diagram is an example of the solution for n even. The second diagram is an example of the solution for n odd.

N. Y. WILSON

18^2 BOARD

	c	a	b	d	c	a	b	d	c	a	b	d	c	a	B	d	
a	b	C	c	a	b	d	c	a	b	d	c	a	b	d	c	a	b
C	d	c	d	a	b	c	d	a	b	c	d	a	b	B	b	B	d
b	a	C	b	c	d	a	b	c	d	a	b	c	d	a	c	b	a
d	c	c	a	C	c	a	b	d	c	a	B	d	B	b	d	d	c
a	b	b	d	a	b	C	c	a	b	d	c	a	b	c	a	a	b
c	d	a	c	C	d	c	d	a	A	B	b	B	d	d	b	c	d
b	a	d	b	b	a	C	b	B	d	a	c	b	a	a	c	b	a
d	c	c	a	d	c	B	a	A	D	A	d	d	c	b	d	d	c
a	b	b	d	a	b	b	C	B	C	c	D	a	b	c	a	a	b
c	d	a	c	c	d	a	c	b	D	d	A	c	d	d	b	c	d
b	a	d	b	b	D	d	D	C	c	b	a	b	A	a	c	b	a
d	c	c	a	d	c	a	b	d	c	a	A	d	c	b	d	d	c
a	b	b	d	D	b	D	C	a	b	d	c	a	A	c	a	a	b
c	d	a	C	b	a	d	c	b	a	d	c	b	a	d	A	C	d
b	D	d	D	d	c	b	a	d	c	b	a	d	c	b	a	b	A
d	C	a	b	d	c	a	b	d	c	a	b	d	C	a	A	d	C
	b	D	c	a	b	d	C	a	b	d	C	a	b	d	C	a	

22^2 BOARD

	c	a	b	d	c	a	b	d	c	a	b	d	c	a	b	d	c	a	B	d	
a	b	C	c	a	b	d	c	a	b	d	c	a	b	d	c	a	b	d	c	a	b
C	d	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	B	b	B	d
b	a	C	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	c	b	a
d	c	c	a	C	c	a	b	d	c	a	b	d	c	a	B	d	B	b	d	d	c
a	b	b	d	a	b	C	c	a	b	d	c	a	b	d	c	a	b	c	a	a	b
c	d	a	c	C	d	c	d	a	b	c	d	a	b	B	b	B	d	d	b	c	d
b	a	d	b	b	a	C	b	c	d	a	b	c	b	a	c	b	a	a	c	b	a
d	c	c	a	d	c	c	a	C	c	a	b	d	B	b	d	d	c	b	d	d	c
a	b	b	d	a	b	b	d	a	b	C	c	a	b	c	a	a	b	c	a	a	b
c	d	a	c	c	d	a	c	c	d	A	D	B	d	d	b	c	d	d	b	c	d
b	a	d	b	b	a	d	b	b	D	B	C	b	a	a	c	b	a	a	c	b	a
d	c	c	a	d	c	c	a	d	c	a	A	d	c	b	d	d	c	b	d	d	c
a	b	b	d	a	b	b	d	D	b	d	c	a	A	c	a	a	b	c	a	a	b
c	d	a	c	c	d	a	c	b	a	d	c	b	a	d	A	c	d	d	b	c	d
b	a	d	b	b	D	d	D	d	c	b	a	d	c	b	a	b	A	a	c	b	a
d	c	c	a	d	c	a	b	d	c	a	b	d	c	a	A	d	c	b	d	d	c
a	b	b	d	D	b	D	c	a	b	d	c	a	b	d	c	a	A	c	a	a	b
c	d	a	c	b	a	d	c	b	a	d	c	b	a	d	c	b	a	d	A	c	d
b	D	d	D	d	c	b	a	d	c	b	a	d	c	b	a	d	c	b	a	b	A
d	c	a	b	d	c	a	b	d	c	a	b	d	c	a	b	d	c	a	A	d	c
	b	D	c	a	b	d	c	a	b	d	c	a	b	d	c	a	b	d	c	a	

GLEANINGS FAR AND NEAR

1943. " Do you know what a mathematician is? " Kelvin once asked a class. He stepped to the board and wrote

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Putting his finger on what he had written, he turned to the class " A mathematician is one to whom *that* is as obvious as that twice two makes four is to you."—(E. T. Bell, *Men of Mathematics.*)

Many things are not accessible to intuition at all, the value of

$$\int_0^{\infty} e^{-x^2}\,dx$$

for instance.—(J. E. Littlewood, *A Mathematician's Miscellany.*) [Per Mr. E. A. Side.]

REVIEWS

Algebra, a Textbook of Determinants, Matrices and Algebraic Forms. By W. L. FERRAR. Second Edition. Pp. 228. 21s. 1957. (Oxford University Press)

Linear Algebra for Undergraduates. By D. C. MURDOCH. Pp. 239. 44s. 1957. (Chapman and Hall)

The first edition of Dr. Ferrar's book appeared in 1941* at a time when mathematical syllabuses in British universities were as yet almost innocent of any taint of algebra, and it may well have contributed to the great change of emphasis that has since taken place. The new edition of the work is to be warmly welcomed. The author has further enhanced its usefulness by adding a chapter on latent vectors.

Whereas in Dr. Ferrar's traditional treatment matrices are regarded as rectangular arrays of numbers which obey certain algebraic rules, Professor Murdoch's approach is uncompromisingly modern. Starting from the notion of a vector space (which, for the sake of simplicity, is always taken as a subspace of the space of complex or real n-tuples), he develops the fundamental concepts of linear mappings of a vector space and of matrix representations of such mappings and then goes on to consider matrices and their simplest canonical forms. In particular, he bases the discussion of orthogonal and unitary matrices and of quadratic forms on a study of vector spaces for which an inner product has been defined. The book is designed as an introductory text for undergraduates and it succeeds admirably in its aim. It does not take the reader very far into linear algebra, but it provides him with all the necessary equipment for a more extensive journey. It could hardly be improved upon either in the choice of material or in the manner of exposition.

L. MIRSKY

A Concrete Approach to Abstract Algebra. By W. W. SAWYER. Pp. 233. 10s. 6d. 1959. (Freeman and Co., U.S.A., Bailey Bros. and Swinfen Ltd., London)

The parts of modern algebra discussed in this book are the field axioms, field extension and polynomials over a field, and vector spaces; the results established are used to prove the impossibility of trisecting an angle by ruler and compass alone, by showing that any " constructible " number lies in a field of dimension 2^n over the rationals whereas the dimension the extension field of a root of the " trisection equation " $w^3 - 3w - 1 = 0$ over the rationals is 3. The approach is concrete in the sense that it proceeds from the familiar to the new; everything that can be done by patient exposition to help the reader *is* done. Anyone familiar with school algebra who has tried—and failed—to read other books on modern algebra will almost certainly not fail with this one, if he conscientiously attempts the examples before looking up the solutions.

* Reviewed in Vol. XXV of the *Mathematical Gazette*.

Only in discussing indeterminates does the author's standard of exposition fall below that of other recent accounts. Sawyer takes the indeterminate x to be the polynomial (0, 1, 0, 0, ...), but it is surely clearer to take an indeterminate over a field $\mathscr{F}$ to be an element of a superfield $\mathscr{G}$ (containing $\mathscr{F}$) which is transcendental over $\mathscr{F}$ (i.e. does not satisfy a polynomial equation with coefficients in $\mathscr{F}$). Moreover, we do not need to introduce the concept of an indeterminate to justify the traditional proof of the remainder theorem; misled by the word division some (like the present reviewer) have mistakenly supposed that division in the sense of cancellation was involved, but in fact so-called long division is just repeated subtraction and is valid not only for polynomials but also for polynomial functions.

The book's value as an introduction would be greatly enhanced by the provision of a good index and bibliography.

R. L. Goodstein

Computational Methods of Linear Algebra. By V. N. Faddeeva. Translated by Curtis D. Benster. Pp. 252. $1.95. 1959. (Dover Publications Inc., New York)

This booklet is divided into three chapters of which the first gives an excellent introduction to the algebra and analysis of matrices: it goes as far as a description of the Jordan form, and a particularly stimulating discussion of norms. The second discusses the problems of the solution of a system of linear equations and the inversion of matrices: the usual direct and indirect methods such as the Gaussian elimination method and the Seidel iterative method are described and compared. The third chapter discusses the determination of the characteristic values and vectors of a matrix. It includes a full discussion of methods for finding the characteristic equation. Throughout the book there are worked examples which have been checked by the translator.

The original Russian edition of this book was published in 1950. In view of the extraordinary recent developments in this field during the last decade the book cannot be entirely up-to-date. Many methods for the inversion and decomposition of matrices have been developed in the last decade and a few of them have been studied, theoretically and experimentally, from the point of view of stability. This book, however, will continue to serve a very useful purpose for researchers, as well as for teachers and students of numerical analysis, because of the clear presentation of the basic facts.

The translation reads well, and the printing is unusually good. The reader will find, in various footnotes to Chapter 1, mysterious references. These refer to a short bibliography which was included in a translation of Chapter 1 prepared at the National Bureau of Standards under the direction of Professor G. E. Forsythe in 1952 but which was left out of the printed version. The following key will help:

1. A. C. Aitken, *Determinants and matrices*, 1948.
2. R. A. Frazer, W. J. Duncan and A. R. Collar, *Elementary matrices*, 1938.

3. H. W. Turnbull and A. C. Aitken, *Canonical matrices*, 1948.
5. C. C. MacDuffee, *Vectors and matrices*, 1943.

OLGA TAUSSKY TODD

Elementary Matrix Theory. By F. E. HOHN. Pp. 305. £2 12s. 6d. 1958. (The Macmillan Publishing Company, New York)

In recent years most publications on linear algebra or matrix theory have exhibited a shift towards a more conceptual treatment of the subject. Little of this tendency is discernible in Professor Hohn's book, which is designed for a class of readers whose familiarity with mathematical ideas is very slight. The pace of the argument, especially in the first half of the book, is intentionally slow, and every new idea is copiously illustrated by worked examples. Indeed, in his anxiety to make each step thoroughly intelligible, the author defers rather too long the introduction of basic concepts which should dominate the development of the subject from the beginning. Thus linear dependence is not defined until p. 123 and vector spaces not until p. 151. In the reviewer's opinion, the drawbacks of this gingerly approach outweigh its advantages; but this is not a point on which it is possible to speak dogmatically. In any case, within the chosen terms of reference, the reader is offered a systematic and lucid account of elementary matrix theory. There is a wealth of exercises and an excellent classified bibliography.

CONTENTS. I. Introduction to matrix algebra. II. Determinants. III. The inverse of a matrix. IV. Rank and equivalence. V. Linear equations and linear dependence. VI. Vector spaces and linear transformations. VII. Unitary and orthogonal transformations. VIII. The characteristic equation of a matrix. IX. Bilinear, quadratic, and hermitian forms.

L. MIRSKY

Aperçu de la Theorie des Polygones reguliers: IV. By PIERRE A. L. ANSPACH. Pp. 301–430. (Privately printed, Bruxelles)

The previous three parts of this work have already been reviewed in the Gazette. The fourth part follows the same lines. It treats mainly of the regular heptagon and enneagon, though other polygons, including the regular 18-gon, make their appearance from time to time. The chief new material relates to curves: conics, a lemniscate, conchoids, strophoids, cardioids, cycloids, and others. There is probably some interesting geometry to be found here, but the author's peculiar notation makes the labour of disinterring it very great. There are a few concessions to the reader in this volume in the shape of summaries and explanations, but the reviewer can only describe the whole in the words which the author himself uses to describe what his work would have been like, had he used cartesian coordinates: "une terre accessible par un labeur superhumain est une terre perdue".

H. M. C.

Einführung in die Funktionentheorie. By I. I. PRIVALOW. Part I. Pp. 163. DM. 7. 1958 Part II. Pp. 194. DM. 8. 1959. (Teubner, Leipzig)

Translated from the 9th Russian Edition by V. Ziegler, the first of these volumes is an outstandingly good introduction to complex function theory—in many respects incomparably good.

This first part deals only with power series, the bilinear transformation, the exponential and logarithmic functions, roots and circular functions. The account of Riemann surfaces is the clearest I have seen in print, and the study of the bilinear transformation is both thorough and illuminating, concluding with an application to Euclidean models of non-Euclidean geometry. I have but one criticism to make; in a book which otherwise shows such clear evidence of careful preparation, the treatment of the argument of a complex number is no better than elsewhere. If arg z is many valued then such an equation as (p. 13)

$$\arg \alpha\beta = \arg \alpha + \arg \beta$$

certainly needs interpretation; presumably it does not express the equality of two numbers, but of two equivalence classes, and cannot be passed over without comment in an introductory text.

Unlike the first volume which concentrates upon a very careful exposition of fundamental notions, giving every assistance to the beginner, the second volume proceeds at a brisk pace and leaves the reader to find out a good deal for himself at some points: for instance that if $f(z)$ does not vanish in a simply connected region then there is an analytic function $g(z)$ such that $f(z) = e^{g(z)}$, or that the reason behind a certain inequality (p. 28) is that a continuous arc of less than unit length drawn on a grid of unit squares enters at most four squares. The ground covered in 200 pages is considerable, and includes Picard's theorem on essential singularities. The best part of the book is the very general treatment of Cauchy's theorem and the author's original work on limiting values of functions defined by Cauchy type integrals. There is a third volume to follow.

R. L. GOODSTEIN

Lectures on Ordinary Differential Equations. By Witold Hurewicz. Pp. 122. 40s. 1958. Technology Press of the M.I.T. (Chapman and Hall, London.)

There is a plethora of books on differential equations. Many of them are made up of a few chapters on formal solutions of equations reinforced by a more or less sketchy account of portions of general theory, in particular of existence of solutions.

The standing of Hurewicz as a mathematician was an assurance that any book of his would be a significant addition to the literature and not just another re-hash of theorems and examples. Hurewicz, a Pole by birth, died in 1956 after an accident in Mexico. He is best known for his fundamental work in dimension-theory and topology. He was at Brown University and at M.I.T. and had given highly successful courses on differential equations.

3. H. W. Turnbull and A. C. Aitken, *Canonical matrices*, 1948.
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$$\arg \alpha\beta = \arg \alpha + \arg \beta$$

certainly needs interpretation; presumably it does not express the equality of two numbers, but of two equivalence classes, and cannot be passed over without comment in an introductory text.

Unlike the first volume which concentrates upon a very careful exposition of fundamental notions, giving every assistance to the beginner, the second volume proceeds at a brisk pace and leaves the reader to find out a good deal for himself at some points: for instance that if $f(z)$ does not vanish in a simply connected region then there is an analytic function $g(z)$ such that $f(z) = e^{g(z)}$, or that the reason behind a certain inequality (p. 28) is that a continuous arc of less than unit length drawn on a grid of unit squares enters at most four squares. The ground covered in 200 pages is considerable, and includes Picard's theorem on essential singularities. The best part of the book is the very general treatment of Cauchy's theorem and the author's original work on limiting values of functions defined by Cauchy type integrals. There is a third volume to follow.

R. L. GOODSTEIN

Lectures on Ordinary Differential Equations. By Witold Hurewicz. Pp. 122. 40s. 1958. Technology Press of the M.I.T. (Chapman and Hall, London.)

There is a plethora of books on differential equations. Many of them are made up of a few chapters on formal solutions of equations reinforced by a more or less sketchy account of portions of general theory, in particular of existence of solutions.

The standing of Hurewicz as a mathematician was an assurance that any book of his would be a significant addition to the literature and not just another re-hash of theorems and examples. Hurewicz, a Pole by birth, died in 1956 after an accident in Mexico. He is best known for his fundamental work in dimension-theory and topology. He was at Brown University and at M.I.T. and had given highly successful courses on differential equations.

A book of 122 pages can cover only a selection of topics. Hurewicz's goal is to expound the natures of solutions of the pair of equations (of dynamical interest)

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y).$$

A first step is to discuss the possible forms of solution when P and Q are linear in x, y. This work is contained in Chapters 4 and 5, and was hitherto not easily accessible—and certainly not in so clear a presentation. Chapters 1, 2, 3 deal with existence theorems and linear systems.

It must be said that this is not a book for anyone below the graduate level. No one who is interested in differential equations at that level should omit to read it.

J. C. Burkill

Introduction to Functional Analysis. By Angus E. Taylor. Pp. xvi. 423. $12.50. 1958. (John Wiley and Sons)

This is the best introduction to the theory of linear operators in the English, and probably in any, language. It begins with an account of the general notions of linear manifolds and operators, and after a chapter in which the necessary topological notions are explained, deals with topological linear spaces, discussing the convex spaces generally but with particular reference to Banach and Hilbert Spaces. The theory of linear operators in these spaces is dealt with in three further chapters, of which two are devoted to spectral analysis in Banach and Hilbert Spaces respectively. A final chapter discusses integration in abstract spaces. The treatment throughout is admirable, both in its clarity and in the selection of subjects for discussion from the abstract theory and its applications to classical analysis. The theory of the text is illustrated and rounded off by numerous examples. The book will be useful to all workers in this field, and especially valuable to students beginning the study of it: for these it has the advantage of requiring no prior knowledge outside the elements of classical analysis and Lebesgue integration.

J. L. B. Cooper

Finite-dimensional Vector Spaces. By Paul R. Halmos. Pp. viii, 199. 37s. 6d. 1958. (D. van Nostrand)

The purpose of this book is to give an account of the theory of linear operators and manifolds in finite-dimensional spaces suitable as an introduction to the theory of operators in Hilbert Space. It adopts an approach based on operator theory, makes little use of matrices and determinants, though these are mentioned, and gives some indication of the use of transcendental methods, such as Zorn's lemma. After an account of operators in general finite spaces, including a rather sketchy account of the Jordan canonical form, the scalar product spaces and the decomposition of hermitian and normal operators in these are treated with great

thoroughness. Finally convergence of operators, with an elementary ergodic theorem, are studied, and a brief account of Hilbert Space follows. The book is written clearly and carefully, and has numerous examples well chosen to illustrate its point of view. It can be recommended strongly for the student of its subject.

J. L. B. COOPER

Integral Equations. By F. SMITHIES. Pp. x, 172. 27s. 6d. 1958. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 49. Cambridge University Press)

This long-awaited tract is intended as a successor to Bôcher's " An Introduction to the study of Integral Equations " (No. 10 in the same series), which has long been out of print. It can be described broadly as an up-to-date version of the latter.

The tract is principally concerned with linear integral equations of the second kind with parameter, such as

$$x(s)=y(s)+\lambda\int_a^b K(s, t)x(t)dt \quad (a\leqslant s\leqslant b).$$

Kernels $K(s, t)$ of two types are considered, to wit continuous kernels and kernels of Lebesgue-integrable square ($\mathfrak{L}^2$ kernels). A distinctive feature of the author's treatment of the latter is that the above equation is required to hold *everywhere*. The usual " identification " of functions equal almost everywhere is not made.

Operators are introduced at an early stage as a notational convenience, but no use is made of any of the deep results of the theory of linear operators. The main prerequisites for reading the tract are a familiarity with matrix theory, and some knowledge of the Lebesgue integral such as can be gained from Burkill's tract (No. 40).

The body of the tract commences with a discussion of the resolvent kernel for regular values of the parameter. Characteristic values and characteristic functions are also considered. Here the author observes the convention (not yet, alas, universally accepted) that a *characteristic value* is a number λ such that the equation $x=\lambda Kx$ has a non-trivial solution whilst an *eigenvalue* is a number κ such that the equation $Kx=\kappa x$ has a non-trivial solution.

The Liouville-Neumann series and a generalization are discussed. The former is applied to Volterra integral equations of the second kind, and is also used in obtaining the Fredholm alternative theorem and determinant-free theorems by approximating to a given kernel by kernels of finite rank.

The classical Fredholm theory is presented for continuous kernels, so far as it concerns the Fredholm determinant and first Fredholm minor—the higher Fredholm minors are not introduced. This theory is then modified so as to be applicable to general $\mathfrak{L}^2$ kernels.

A discussion of Hermitian kernels includes expansion theorems in terms of characteristic functions and characteristic values, and extremal properties of characteristic values. Here, and elsewhere, use is made of the theory of orthonormal systems (to which a chapter is devoted).

Finally, these expansions are generalized to non-Hermitian kernels by introducing singular values and singular functions. These are used for the study of normal kernels, and also applied to linear integral equations of the *first* kind.

The tract is presented in the author's usual lucid style, and is a pleasure to read. It is well supplied with references for anyone who wishes to delve deeper into the theory or its applications than is possible within the limits of a tract. The printing is good, and the reviewer has detected only one misprint of any importance—on line 6 of page 108 where an exponent " 2^{r-1} " has been omitted from the kernel.

A. F. RUSTON

Gitterpunkte in Mehrdimensionalen Kugeln. By A. WALFISZ. 1957. (Warsaw)

This book is devoted to the study of the function

$$P_q(t) = A_q(t) - V_q(t),$$

where $V_q(t)$ is the volume of the q-dimensional sphere with radius t and centre at the origin and $A_q(t)$ is the number of lattice points (*i.e.* points all of whose coordinates are rational integers) within or on the surface of this sphere. The author's style of exposition is as limpidly clear as Landau's and the book could be read with especial profit by anyone commencing work as a mathematical author. The beautiful theory is due to a sequence of celebrated mathematicians, including Landau, Hardy, Jarnik, Vinogradoff, van der Corput and, most of all, Professor Walfisz himself. The book includes a detailed account of sources and a brief account of results in the more general problem for ellipsoids.

E. M. WRIGHT

Asymptotic Methods in Analysis. By N. G. DE BRUJIN. Pp. 200. 20 guilders. 1958. (North Holland Publishing Co., Amsterdam)

This book is designed " to teach asymptotic methods by explaining a number of examples in detail, so as to suit beginners who seriously want to acquire some technique in attacking asymptotic problems ". The author wisely avoids formulating a general theory. This is a subject in which general theorems are almost always cumbrous, difficult to understand and tedious, whereas the methods, especially when applied to particular examples, can be simple, powerful and elegant (and certainly are all three, when expounded by Professor de Brujin). Again there is no attempt to list *results* and for a bibliography we are referred to Erdelyi's book on *Asymptotic Expansions*.

The titles of the chapters are 1. Introduction (see p. 19 for a very illuminating conversation between a Numerical Analyst and an Asymptotic Analyst), 2. Implicit Functions, 3. Summation, 4. The Laplace method for integrals, 5. The saddle-point method, 6. Applications of the saddle-point method, 7. Indirect Asymptotics, 8. Iterated functions,

9. Differential equations.

How pleasant it will be to have a satisfactory answer when I am next asked where an introductory account of the saddle-point method can be found.

E. M. WRIGHT

Approximate Methods of Higher Analysis. By L. V. KANTOROVICH and V. I. KRYLOV. Translated from the 3rd Russian Edition by C. D. Benster. Pp. 681. 1958. (Groningen, Noordhoff)

This book is concerned with approximate methods used in the solution of partial differential equations, conformal mapping and the approximate solution of integral equations. The titles of the chapters are: I. Methods based on the representation of the solution as an infinite series, II. The approximate solution of the integral equations of Fredholm, III. The method of nets, IV. Variational methods, V. The conformal transformation of regions, VI. Principles of the application of conformal transformation to the solution of the fundamental problems for canonical regions, VII. Schwarz's method.

The publisher's note says that the book is translated from the third Russian edition, which appeared in 1950 (according to *Mathematical Reviews*); there is no indication of this date in the volume under review. The translator says that he has followed the fourth Russian edition, of which he gives no date and which I cannot trace in *Mathematical Reviews*. Thus the " approximate methods " seem to extend beyond the mathematics. The translation on the other hand occasionally has a literal flavour. None of these trivial criticisms however detract from the very great usefulness of a translation of a standard text in a field to which Russian mathematicians have contributed largely.

E. M. WRIGHT

Ordinary Difference-Differential Equations. By E. PINNEY. Pp. 262. 37s. 6d. 1958. (University of California Press)

The bulk of this book is concerned with what the author calls " mixed " difference-differential equations, i.e. equations connecting

$$y^{(n)}(t), \ldots, y(t), y^{(n)}(t-b_1), \ldots, y^{(n)}(t-b_m), \ldots, y(t-b_m).$$

This is the first book on the subject in English. There is a massive and very scattered literature and the task of giving an adequate and coherent account of the somewhat scrappy theory would be formidable. The author does not attempt this but presents his own approach to (i) simultaneous linear equations with constant coefficients and (ii) the non-linear equation. With differential equations, there is an immediate gain in considering a family of *simultaneous* linear equations, since one can confine oneself, without loss of generality, to equations of the first order. No similar gain is possible here since, at the best, one would have equations of first differential order and of first difference order. The author does not use matrices, and so his consideration of simultaneous equations leads to very elaborate formulae (occasionally one formula covers half a page).

The author is primarily concerned with applications and does not take rigour very seriously. Unfortunately he extends this to the use of complex integration, where such an attitude is highly dangerous. Thus on p. 34 he does not shrink from taking a contour through the poles of the integrand (though this does little harm!), while the main Theorems 2.1 and 2.3 are partly false. The series solutions may diverge, if one does not group the terms suitably. A paper by Leont'ev, to which the author refers in another connection, is partly concerned with constructing a Gegenbeispiel to show this. Papers by Pitt and others, to which the author also refers, also make the point clear. I have some sympathy with the author here for I once made precisely the same error in *quoting* a result. But, if one tries to write out a rigorous proof, the point becomes obvious.

A large part of the book is devoted to the detailed application of the general theory to particular examples. It would be interesting to know whether any of these particular results have been tested in practice. The author has overlooked the two pre-war papers of Hartree, Callendar, etc. These authors discussed a particular example, solved it on the differential analyser and built a control mechanism to illustrate the results.

The bibliography is lengthy but inevitably incomplete. The author specifically disclaims any intention to give a full survey. This does not greatly matter, however, since the reader who wants a masterly sketch of the literature and a first class bibliography can go to Hahn (Bericht über Differential-Differenzengleichungen mit festen und veränderlichen Spannen, *Jahresber. d. Deutscher Math. Ver.* 57 (1954), 55–84).

E. M. Wright

An Introduction to the Theory of Measure and Integration. By A. C. Zaanen. Pp. 254. 50s. 1958. (North-Holland Publishing Co. Amsterdam)

Writers on measure and integration nowadays are divided into two schools of thought: those who define measure in terms of integration, and those who define integration in terms of measure. Professor Zaanen's aim in this book is, within the limitation on length imposed by the requirement that the book is to be used as a university text, to introduce the reader to both schools, and the reviewer believes he has succeeded. First he shows that under suitable conditions a measure defined on a σ semi-ring Γ of subsets of a set X can be extended to a countably additive measure on the σ-ring Γ_1 generated by Γ, and establishes some of the classical properties of the extended measure. Next, he uses these results to obtain a corresponding theorem on the extension of a suitable linear functional $I(f)$ defined on a linear collection of real functions on X to a linear functional (a Daniell integral) on a larger linear collection M; if in particular X is a measure space and L consists of measurable step-functions, the extended integral I is a Lebesgue-Stieltjes integral. From this basis the fundamental properties of the integral are derived, after which optional topics and applications are

treated. The later chapters cover a wide variety of subjects: Fubini's theorem, the Radon-Nikodym theorem, differentation, change of variable, measure on the line, signed measures, ergodic theory, Banach spaces, conjugate spaces, and unitary transformations in Hilbert space.

The book makes excellent reading. It is clearly written and admirably sign-posted. Despite the surprising amount of material covered in a short work, it is not over-condensed. Some of the brevity is obtained by the now popular device of leaving many of the important theorems as exercises for the reader. (For instance, Egoroff's theorem, the theorem that a monotonic function on the line has a derivative almost everywhere, and much of the theory of product spaces occur in this form; the reviewer was unable to find Lusin's theorem anywhere.) The theory of Lebesgue measure in Euclidean n-space is not treated systematically, but is covered partly by being used as the principal illustration and example, and partly in the exercises. This constant reference to Euclidean space saves the book from being very abstract for a beginner.

In the reviewer's opinion, Professor Zaanen's approach is a little abstract and a little devious for the beginning student, who might well profit more by a straightforward course on Lebesgue theory; nevertheless, the book is a useful contribution to the literature.

D. M. STONE

Electronic Digital Computers. By F. L. ALT. Pp. 336. $10. 1958. (Academic Press, New York)

This book provides an extremely good introduction to the subject of electronic digital computers, both to those who are going to prepare problems for them, and to those who are going to be concerned with their design and construction. The book is made up as follows. Part 1 (2 chapters) is an introduction and historical survey—which curiously makes no mention of the work of Charles Babbage, and the only reference to developments prior to 1940 is the remark " Earlier pioneering efforts seem to have had no lasting influence"! Part 2 (6 chapters) is an account of the principles underlying the logical design and physical construction of automatic computers. Part 3 (4 chapters) deals with programming. This is a good account of general principles illustrated where necessary by reference to SEAC, and the I.B.M. 704, the machines at the National Bureau of Standards, with which the author is personally familiar. Part 4 (8 chapters) deals with the numerical mathematical techniques encountered in scientific and engineering type calculations. The book is worth reading for this section alone which is tantamount to a handbook of numerical analysis. Part 5 (3 chapters) is entitled " Matching problems and machines ". The first of these last three chapters is a survey of scientific problem areas, e.g., astronomy, fluid mechanics, meteorology, optics, etc. Here the author is able to draw on his experience at N.B.S., to which a very wide variety of problems are submitted. The second chapter deals with problems from other areas. These include business type calculations, where the mathematics involved is usually elementary, but the amount of data to be handled is very large; and real time process

control where the computer is part of a larger system, e.g., control of a milling machine, or a chemical process. The last chapter deals with the organisation and staffing of a computer installation. Here again the author is able to draw on his experience with N.B.S., and it is interesting to note that he recommends the " closed shop " system of programming, i.e., that all problems be handled by a central staff, in contrast to the " open shop " system (practised by most University Computer Centres with a limited staff) where customers are encouraged to do their own programming. There is a useful bibliography at the end of the book.

R. A. BROOKER

Contributions to the Theory of Nonlinear Oscillations, IV. (Annals of Mathematics Studies, No. 41.) Edited by S. LEFSCHETZ. Pp. 211. 30s. 1958. (Princeton University Press)

Like the other three volumes this contains a number of new papers on the subject including one by S. Kakutani and L. Markus on a nonlinear difference-differential equation. W. T. Kyner follows up his work with S. Diliberto in volume III, and R. de Vogelaere deals with the periodic solutions of Størmer's problem which arises in electro-magnetic theory. In " Optimal Discontinuous Forcing Terms " D. Bushaw deals with the rapidity with which the origin is reached when the forcing term of a second order equation only takes the values $+1$ and -1, and he determines the best way in which to make the forcing term change sign. This is a problem of control theory which Bushaw has discussed more generally in a recent mimeographed report. However, the paper of most general interest is probably " A Survey of Lyapunov's Second Method " by H. A. Antosiewicz with a long list of references including many Russian ones. It is unfortunate that V. I. Zubov's book *The Methods of A. M. Lyapunov and their applications* (Leningrad University Press 1957) came out too late for inclusion in the references to Zubov's work.

M. L. CARTWRIGHT

Foundations of Set Theory. By A. A. FRAENKEL and Y. BAR-HILL. Pp. 415. 84s. 1958. (North Holland Publishing Co.)

Written for graduate students in mathematics and philosophy this masterly survey of the foundations of mathematics covers not only set theory but the whole range of foundation studies. After a vivid historical introduction describing the paradoxes of set theory there follows an account of the axiomatic foundations of the Zermelo style system which Fraenkel used informally in his introductory volume in the same series, *Abstract Set Theory*. A central feature of this account is a very full discussion of the axiom of choice. The authors employ a balanced blend of formal and informal methods of presentation, symbolic logic being introduced mainly in the statement of axioms and definitions, to ensure accuracy. This formal system is then compared with systems which have the closest affinity to it, the systems associated with the names of von Neumann, Bernays and Gödel. Next follows an account of systems

which employ type theory with special reference to Quine's *New Foundations* and *Mathematical Logic*, and also some account of Lorenzen's and Wang's recent systems.

The scope of the book broadens in the fourth chapter where there is an illuminating presentation of intuitionist and allied ideas. This excursion into intuitionism is noteworthy for the impartiality with which the case for intuitionism is presented. The final chapter is devoted to Gödel's incompleteness theorem and current work which finds its origin in Gödel's theorem. The account of Gödel's work is however just a sketch and the authors frankly refer the reader to other accounts for details.

Like the earlier *Abstract Set Theory* the book concludes with a most impressive and valuable bibliography of publications in the field (chiefly devoted to the period 1950–7).

For its genial style and its appraisal of new work as well as for its accumulation of historical detail this book will be studied with pleasure and profit by a wide circle of readers.

R. L. GOODSTEIN

Special Relativity for Physicists. By G. STEPHENSON and C. W. KILMISTER. Pp. 108. 18s. 1958. (Longmans, Green and Co. Ltd.)

The main purpose of this book is to give an account of the special theory of relativity without making heavy demands on the mathematical abilities of the reader. This is achieved by not using tensor calculus, although this leads to rather clumsy expressions for the transformation properties of four-vectors and six-vectors and the structure of the equations is not so easily seen. The reader is assumed familiar with three dimensional vector calculus and the extra mathematics required is dictated mainly by each particular application.

After a brief discussion of the difficulties of pre-relativity physics, Lorentz transformations are considered from several points of view and this is followed by an account of their direct consequences. More than half of the book is devoted to applications. These range over a very wide field, including many effects in modern physics for which an understanding of special relativity is required. The Authors have succeeded in most cases in giving a concise account of the principles involved. Many references are given to other treatments or extensions of the theory, and these greatly enhance the value of the book.

This book will be welcomed by anyone who wants to know about the special theory of relativity or its applications without having to understand tensor calculus first.

A. WEINMANN

Theory of Relativity. By W. PAULI.

This is a translation of Pauli's well-known *Encyklopädie* article of 1921, with supplementary notes written by him in 1956. There is probably still no survey of relativity theory to rival it. Its benefit can be had by anyone possessing only a rudimentary previous acquaintance with the subject, as the reviewer found from reading the original version

many years ago. Re-reading it now shows one how comparatively little fundamental progress has been made since the early days. This is because so much of the subsequent effort has gone into attempts to produce unified field-theories, which for reasons cogently stated by Pauli would not be expected to succeed, and into applications to cosmology, which have mostly involved no very fundamental problems in the theory itself. However, attention has gradually come to be re-focused upon the most basic questions of interpretation of the theory and, at the present time, these questions are being attacked with renewed vigour. The re-publication of Pauli's work is most timely.

W. H. McCrea

Corpuscules et Champs en Théorie Fonctionnelle. [Les Grands Problèmes des Sciences, IX.] By Jean-Louis Destouches. Pp. 161. 4,000 Fr. 1958. (Gauthier-Villars, Paris)

Professor Destouches proposes to elaborate the treatment of fundamental and elementary particles by means of a functional representation. The set of functions representing a particle of a specified type is to satisfy a set of differential equations composed of linear parts given by standard wave-mechanics together with new non-linear parts. For practical purposes the book can be described as a survey of what might be achieved by admitting the presence of these non-linear terms. After treating standard fundamental particles, the author considers photons and non-linear electrodynamics and then " gravitons " and non-linear gravitational theory. He sees in his method a possible approach to a unified theory of electromagnetism and gravitation. However, the whole treatment seems to be highly tentative and it is not apparent what general principles are to be followed in making it more precise, nor how such a treatment could include a theory of space-time.

W. H. McCrea

Abelian Groups. By L. Fuchs. Pp. 367. 64s. 6d. 1958. (Budapest, Hungary)

It takes courage to write a treatise on abelian groups so soon after the appearance of Kaplansky's lively little book. It takes courage also to write such a book in a language not one's own. The book under review is a testimony not only to the author's great courage, but also to his tremendous—and infectious—enthusiasm. There are small blemishes, a minor misprint here, a slight idiomatic oddity there, but all this is insignificant compared to the many excellent features of the book. Professor Fuchs has packed a very great amount of important and interesting group theory into it; he always explains what he intends to do, and why; he not only proves theorems but also discusses their significance. The vast amount of material selected for presentation includes everything that is of real importance; many of the lesser results are relegated to the exercises that accompany every chapter. There are some 550 illustrative exercises in all, many with hints for solution, others

with asterisks to warn of special difficulty. Many unsolved problems are formulated. Numerous careful references, a comprehensive bibliography, indices of authors, subjects, and notation contribute to make the book as useful a work of reference for the initiate as it is an inspiring introduction for the novice.

B. H. NEUMANN

Gruppentheorie. By L. BAUMGARTNER. Third Edition. Pp. 110. DM. 2,40. 1958. (Walter de Gruyter and Co., Berlin)

This well-known little book has been completely revised for the new edition. It is quite astonishing how much is contained in 110 small pages in a form which is still highly readable. There are many worked examples and problems with solutions and though perhaps not suitable for beginners, this makes an ideal revision course in the elements of group theory; in addition to " pure " group theory, there are sections on homomorphy and isomorphy and a section on free groups. The penultimate paragraphs are devoted to decreasing sequences of groups with a proof of Schreier's Theorem; this is followed by the Jordan-Hölder Theorem, and an appendix on alternative group postulates.

R. L. GOODSTEIN

Studies in Linear and Non-Linear Programming. (Stanford Mathematical Studies in the Social Sciences.) By K. J. ARROW, L. HURWICZ, and H. UZAWA. Pp. 229. 60s. 1958. (Stanford Univ. Press)

The three main authors, supported by five of their colleagues, produce and consolidate recent advances in the field of mathematical programming. The first chapter, " Introduction ", contains a summary and orientation, while the remaining 14 chapters, grouped in three parts, deal with existence theorems, the gradient method, and with other methods of linear and non-linear programming. The first two parts are written for mathematicians, and the third is addressed to economists.

S. VAJDA

Elementary Mathematical Programming. By R. W. METZGER. Pp. 246. 48s. 1958. (Chapman and Hall)

Books on mathematical techniques of Operational Research are now being published at such a rate that it is as well to describe their scope and the readers which they wish to attract. The present volume deals, apart from a few non-mathematical asides (such as on pp. 218 ff. about round-trip schedules) with Linear Programming, and with no other branch of what is now called Mathematical Programming. The adjective " elementary " in the title is justified, and the book succeeds admirably in its aim to bridge the gap between non-technical and mainly descriptive texts on the one hand and highly technical presentations on the other.

The choice and succession of material is the usual one. After a short Introduction, Chapter 2: Distribution Methods, deals with the "stepping stone method" of solving the Transportation Problem, and with an approximation method (which the reviewer finds over-praised). Chapter 3: The Simplex Method, is the longest of the book, as befits the importance of the subject matter. "The dual formulation . . . is beyond the scope of the book" (p. 112). This is a pity, because that formulation supplies information of great practical value ("shadow prices"). Chapter 4: Approximation Methods, mentions a number of them, but the reviewer has not sufficient experience to comment on their value. Chapter 5: Typical Problems and Their Solution, gives two illustrations, while Chapter 6: Computers and Mathematical Programming, is of doubtful value. On merely five pages, it lists computer programs, all of the same manufacturing firm.

The remaining chapters contain more illustrative examples of applications, somewhat similar to those in the reviewer's Readings in Linear Programming. It is surprising that obvious references are not made, e.g. to K. Eisemann's The Trim Problem, in connection with Chapter 8: Stock Splitting (the same thing) or to Charnes and Ferguson's Optimal Estimation of Executive Compensation by Linear Programming, in connection with Chapter 10: Job and Salary Evaluation. (Both these earlier papers appeared in Management Science, vols. 3 and 1 respectively.)

The style of presentation is so splendid, that it could almost be thought to prove the possibility of teaching mathematics without a single proof. However, pp. 112–117 on the Modified Simplex Method, with their flavour of a cookery book, confirmed your reviewer's doubts on this score.

The bibliography is divided into three parts: Relatively Easy Reading, References for the More Mathematically Inclined, and Technical References, with some overlap. The Index is very poor.

The above paragraphs concentrated, we hope not unfairly, on some shortcomings. However, this is a very good book, not too highbrow and not too lowbrow. It is thoroughly to be recommended as reading for all levels, from the manager down to those back-rooms where the sums are actually done.

S. Vajda

Smoking—the Cancer Controversy. By Sir Ronald A. Fisher. Pp. 47. 2s. 6d. 1959. (Edinburgh. Oliver and Boyd)

The publishers describe this short collection of letters and lectures as "a fair-minded assessment of the value of the statistical evidence relating to the incidence of lung-cancer in smokers". Apart from one of the lectures, which summarises the author's well-known views on the nature of probability and the processes of scientific inference, the pamphlet is an attack on the "orthodox" view that smoking causes lung cancer. The main argument is the unexceptionable statement that statistical correla-

tion and causal connection are not synonymous, and two possible alternatives to a causal connection between smoking and cancer are put forward. The first, that incipient cancer causes smoking, is not at present a serious contender, and Sir Ronald produces no evidence for it. The second, that cigarette smoking and cancer are both influenced by the genetic structure of the individual, deserves more serious consideration.

The evidence for this latter hypothesis consists of two studies of the smoking behaviour of identical twins as compared with that of non-identical twins. In both cases the identical twins exhibited significantly greater similarity of behaviour than the others, and this remained true if the twins had been separated at birth. Although it appears probable, therefore, that smoking habits are, at least partially, determined by genetic structure, this is a long way from proving the genetic hypothesis concerning lung cancer. No other evidence has yet been forthcoming that lung cancer and genotype are linked, although there is analogous evidence in other diseases.

Sir Ronald's evidence against the causal theory is also rather thin. He notes that the increase in lung cancer has been preponderantly among men, " but it is notorious, and conspicuous in the memory of most of us that over the last fifty years the increase of smoking among women has been great, and that among men . . . certainly small." This may be misleading. If, as is claimed by adherents of the causal hypothesis, it is consistent heavy smoking that does the damage, the rate of increase in women smokers is irrelevant until they become heavy smokers. Whether a substantial number of women have been heavy smokers for any length of time is not a matter which can safely be judged by memory.

His other major point is equally inconclusive and concerns inhaling. (It is noted in passing that Table 2 on page 46 should be disregarded; it is, at best, meaningless and, at worst, misleading.) Sir Ronald points to the fact that smokers who inhale appear to have a lower incidence of lung cancer than smokers who do not. Even if the causal hypothesis were a true explanation of the statistical connection, the mechanics of the causative process are certainly unknown, so that a difference in effect between two different techniques of smoking should cause no surprise. And it is untrue to suggest, as Sir Ronald does albeit sarcastically, that the statistical research workers have " discovered the means of its [lung cancer's] prevention (inhaling cigarette smoke) ". An examination of the data given in the relevant section of the pamphlet shows that, if inhalers among cancer patients are compared with inhalers among controls, there remains a significant preponderance of heavy smokers among the former.

On the existing evidence both the causal and the genetic hypothesis appear to have approximately equal status, although the causal hypothesis, being simpler, has precedence. Also it should be pointed out that the genetic hypothesis does not, in itself, explain the observed increase in lung cancer. Either an increase in the genotype susceptible to cancer, or some other external cause such as air pollution or, indeed, smoking itself, must still be added. Thus, both hypotheses, or neither, could still easily be true.

In a subject where emotional involvement is high, a contributor should, perhaps, declare an interest; your reviewer admits to smoking, and inhaling, approximately twenty cigarettes a day.

F. DOWNTON

Multivariate Correlational Analysis. By P. H. DUBOIS. Pp. xv, 202. 36s. 1957. (Harper and Bros., New York; Hamish Hamilton, London)

An Introduction to Multivariate Statistical Analysis. By T. W. ANDERSON. Pp. xii, 374. 100s. 1958. (John Wiley, New York; Chapman and Hall, London)

Some Aspects of Multivariate Analysis. By S. N. ROY. Pp. vii, 214. 64s. 1957. (John Wiley, New York; Indian Statistical Institute, Calcutta; Chapman and Hall, London)

To read these three books is an instructive reminder of the different levels that exist in the statistical hierarchy. Although neither mutually exclusive nor exhaustive, three of these are the user, the teacher and the mathematical research worker.

Professor DuBois, a psychologist, has written a user's book and it is mainly devoted to the techniques, by which correlations and partial correlations may be computed. Not that this is a book for the beginner, for the opening paragraph of Chapter 1 reads:

"A coefficient of multiple correlation, $R_{0(12 \ldots n)}$, indicates the degree of relationship between 2 (and only 2) variables: an unmodified variable, X_0, and a composite variate consisting of the weighted sum of 2 or more variates, $(X_1, X_2, \ldots, X_n)$, each of which is so weighted that the Pearson product-moment correlation between X_0 and the composite is as high as possible."

The preface gives no indication of the qualifications required of a reader, but the paragraph quoted above shows the need for familiarity with (random) variables or variates—Professor DuBois explains rather irritatingly that he uses the words interchangeably throughout—and with two-variable product-moment correlation. An acquaintance with the method of least squares (mentioned on page 2 without explanation and given a very cursory discussion in Chapter 13) would also seem desirable, as well as a general knowledge of many other parts of elementary statistical theory.

It is unjust to imply that Professor Anderson's volume is only a book for the teacher of graduate students. It is a great deal more than this. He has made the mathematics of multivariate analysis appear straightforward and elegant and this is no mean achievement. Most mathematical statisticians will wish to see this book on their shelves for reference in this branch of the subject.

Starting from the opening chapters on the multivariate normal distribution, the theory is developed systematically through maximum likelihood estimation of the mean vector and variance matrix, sampling distributions, the T^2 statistic, the Wishart distribution and various significance tests to a discussion of principal components, canonical

correlations and the distribution of certain characteristic roots. Since matrices are used throughout there is an appendix on matrix theory, followed by an excellent bibliography.

Professor Roy's book is a much more specialised volume, for, as he says, " this monograph is primarily concerned with the developments in multivariate analysis, in which the author has been specially interested." Although the problems considered here are basically the same as those discussed in Professor Anderson's book, the solutions have little in common, for Professor Roy has an individual approach to the construction of significance tests, which leads in a rather different direction from that taken by the more normal likelihood ratio approach. The final chapter introduces the relatively new idea of treating contingency tables by non-parametric multivariate analysis. Professor Roy feels that this approach is " more realistic and physically meaningful " than, and will ultimately supplant, the existing techniques of analysis of variance and multivariate analysis. This is a fascinating book and statistical research workers will welcome it. That it is, however, very heavy going may be deduced from the fact that it requires nine fairly long mathematical appendices to provide results for use in the main text.

F. Downton

Formeln und Tabellen der mathematischen Statistik. By V. Graf and H. J. Henning. Pp. vii, 104. D.M. 12.60. 1958. (Springer-Verlag)

In this book, largely aimed at the technologist, the process of turning statistical theory into a series of recipes has been carried to its logical conclusion. 31 pages of formulae and 26 pages of examples of their use are followed by 33 pages of tables. The tables include the ordinates of the normal distribution, values of the normal integral together with some percentage points, the 95%, 99% and 99.9% points of the t-, F- and χ^2- distributions, five tables of constants for use in constructing control charts and also values of log $n!$, n^2 and $\sqrt{n}$. Finally there are 12 charts, mostly for determining confidence intervals. The excellent printing and binding make the price rather high for those who only require a book of elementary statistical tables.

F. Downton

Combination of Observations. By W. M. Smart. Pp. xiv, 253. 35s. 1958. (Cambridge University Press, London)

Written mainly for astronomers by an astronomer this book invites comparison with Whittaker and Robinson's Calculus of Observations, first published in 1924. Apart from the similarity of title and style, however, Professor Smart's purpose is more modest than was the earlier authors'. He is concerned only with the statistical theory of errors, and not with that part of the subject which has come to be known as Numerical Analysis. Within this restricted field the present volume provides a

very detailed discussion, although the use, for example, of probable errors has a slightly archaic ring about it. A statistician must deplore the apparent failure of statistical thinking over the past quarter of a century to make any impression on at least one group of scientists. (The only statisticians referred to by the author are Gauss, Laplace and Karl Pearson.) It may be that a more sophisticated approach is unnecessary for astronomers, and within the chosen limits the presentation is clear and thorough, if a trifle dogmatic.

The contents include: frequency distributions, the method of least squares, probability, the Normal law, measures of precision, equations of condition (the least squares solution of n equations in m unknowns when $n > m$), theoretical frequency distributions, correction of statistics and correlation; finally there are four short tables, including one of erf t.

F. Downton

Studies in the Mathematical Theory of Inventory and Production. By K. J. Arrow, S. Karlin and H. Scarf. Pp. x, 340. 70s. 1958. (Stanford University Press; O.U.P., London)

All but one of this excellent collection of seventeen research papers was written, at least in part, by the named authors, and all these papers are here published for the first time. The volume is concerned with various mathematical models which have been put forward as representing storage systems, and the papers range from a historical survey, through optimal policies in both deterministic and stochastic storage systems, to more detailed consideration of the properties of the models for these and analogous systems.

It may be remarked that the previous occasion, on which mathematicians and economists showed interest in this subject, was in the decade immediately preceding the depression of the 1930's. It is to be hoped this was a coincidence.

F. Downton

Finite Queuing Tables. By L. G. Peck and R. N. Hazelwood. Pp. xvi, 210. 68s. 1958. (John Wiley, New York; Chapman and Hall, London)

The type of problem, for which these tables offer a solution, is as follows: N automatic machines each stop running independently and at random instants of time, the average rate of breakdown being $1/U$ per machine. M operators look after these machines and the time taken for an operator to restart a machine has an exponential distribution with mean T. $X = T/(T+U)$, the service factor, is a measure of the reliability of the machine. A system defined in these terms may be regarded as a finite queue (of machines waiting to be serviced) and the properties of this queue in statistical equilibrium are completely determined by N, M and X.

The quantities tabulated here (to 3 decimal places) are D, the probability that in equilibrium a defective machine may have to wait before

repairs can start because no operator is available, and F, the efficiency factor. $F = (T+U)/(T+U+W)$, where W is the average time a machine must wait for repairs to start, so that F is a measure of the amount of use obtained from a machine.

The ranges of N and X are:

$$N = 4(1)26(2)70(5)170(10)250$$

$$10^3 . X = 1(1)26(2)70(5)170(10)340(20)600(50)950.$$

For smaller values of N and X only those values of X, where F changes by one in the third decimal place are given. Values of M appear to range in unit increments from the smallest value, for which D differs from unity to the accuracy of the tables, up to the largest value, for which F differs from unity to the same accuracy. The tables were computed on the UNIVAC 1 computer, and produced by photo-offset from the machine printing, so that no human editing was required.

From the tables other quantities (for example, W the average waiting time) may readily be calculated if required. The model described here, however, represents only the simplest form of machine interference problem (or other finite queuing situation) and the tables will often provide only a rough but very useful guide to the type of solution to be expected in a practical case.

F. Downton

Logique Mathematique Appliquée. By H. Freudenthal. Pp. 57. 1,200 Fr. 1958. (Gauthier-Villars, Paris)

This is an introductory text in the first instance but the thoughts expressed in Chapter 3 are of interest to all who study the *significance* of mathematical logic.

There is a brief but comprehensive account of truth functions, with a proof that there are only two Sheffer functions of two variables, and the formal structure of an electrical binary adder is analysed in logical terms.

Discussing the influence exerted by mathematical logic on philosophy Freudenthal points out (p. 31) the danger of over-reliance upon some of the devices which modern logicians have tended to overwork, like the use of quotation marks for distinguishing *use* and *mention*.

R. L. Goodstein

Topological Analysis. By G. T. Whyburn. Pp. ix, 119. 32s. 1958. Princeton Mathematical Series No. 23. (Princeton University Press; Oxford University Press)

" Topological analysis consists of those basic theorems of analysis, especially of the functions of a complex variable, which are essentially topological in character, developed and proved entirely by topological and pseudo-topological methods " (from the Introduction).

Inter alia the book contains (1) careful proofs of the Jordan curve theorem and of other basic theorems in plane topology, (2) a wealth of classical results in the topology of abstract spaces, (3) an excellent account of the modern theory of " light " and " open " mappings, both for abstract spaces and for the plane, (4) applications to analytic functions of a complex variable.

For (1) and (2) I have nothing but praise. I have already described (3) as excellent: but I did find it difficult reading, and while my difficulties were mainly the result of mental sclerosis I think that the author has been rather severe here. For instance, in order to appreciate a theorem that A and B together imply C I found it helpful to have before me a counter-example showing that C does not follow from A alone: this illuminated both the meaning of B and its relevance to C. The book gives no such examples: and their construction is seldom obvious.

Of (4), however, I feel quite critical. In the first place, it seems to me to have the opposite fault to that I have found in (3). Not a few pages, occupied with such familiar topics as the definition of complex numbers, go very slowly and are to my mind quite unnecessary in a book of this level: I would prune twenty pages or so here and use them to be more explanatory elsewhere. In the second place, the scale of values seems to me all wrong. It is hard to resist the conclusion that great virtue is seen in purely topological proofs of theorems of complex function theory. I am *over*-emphasising my point, but it is a true one, and the attitude I criticise is shared by some other workers in this field beside Professor Whyburn. To an unscrupulous person like myself, all proofs are good but the shortest and easiest is the best: and if the result is " essentially topological " but the proof untopological, or vice versa, this only adds a certain relish. Thus I see virtue not in purely topological proofs of theorems of complex function theory but in proofs (more or less inevitably topological) of topological theorems which *generalize* those theorems of complex function theory.

Having thus bared my own soul, let me say that, even with what I regard as its faults, this is a very valuable book.

H. D. Ursell

An Introduction to Combinatorial Analysis. By J. Riordan. Pp. 244. 68s. 1958. (Chapman and Hall)

This is a very useful collection of results on combinations and permutations and the construction of generating functions. The book presupposes no technical knowledge of mathematics beyond a competence in school algebra, and achieves a great deal on this limited basis. There is more here about the cycles of permutations than is to be found elsewhere and also some new work on trees and linear graphs.

R. L. Goodstein

Introduction a L'Algèbre Supérieure et au Calcul Numérique Algébrique. By L. Derwidué. Pp. 432. 6,600 Fr. 1957. (Masson, Paris)

The fundamental aim of this book is to describe in detail some practical methods for solving linear and polynomial equations, together with the underlying theory. There is a very clear and illuminating account of matrices and determinants, including the Cayley-Hamilton Theorem, and the treatment is elegant without being oblique. The section on the root squaring method for finding roots of polynomials has an historical

introduction which shows Dandelin as the original discoverer and Lobatchevsky and Graeffe as runners up. Another topic studied in detail is the Routh and Hurwitz conditions for a polynomial to have all its zeros in a given half plane. The last chapter gives a brief but valuable account of groups, rings and fields.

R. L. GOODSTEIN

Space and Time. By HANS REICHENBACH. Pp. 295. $2.45. 1958. (Dover, New York; Constable and Co., London)

This translation of Reichenbach's profound and lucid *Philosophie der Raum-Zeit-Lehre* (published in 1928) by the author's wife Maria and by J. Freund, has a brief foreword by R. Carnap, which describes Reichenbach's original publication as a " landmark in the development of the empiricist conception of geometry ".

R. L. GOODSTEIN

The Computer and the Brain. By J. VON NEUMANN. Pp. 82. 24s. 1958. (Yale University Press; Oxford University Press, London)

When one of the greatest mathematicians of our times died in February 1957 he left unfinished the manuscript of the Silliman Memorial Lecture which he had been invited to give in the Spring Term of 1956, and in honour of his memory the Silliman Lecture Committee have published these preparatory notes in the series of Silliman Lectures Publications.

These notes consist of a non-technical account of digital and analogue computers, their use and nature, and a comparison of computers with the brain in relation to speed and size. A moving preface by von Neumann's widow briefly traces his career from his birth in Budapest in 1903 to the onset of bone cancer in 1955.

R. L. GOODSTEIN

BRIEF MENTION

Exercises in Elementary Mathematics. By K. B. SWAINE. Book 1 Revised Edition. Pp. 183. 5s. 6d. 1959. (Harrap)

The revised edition contains more problems on the addition and multiplication of fractions and an amplified account of decimal manipulation.

Intermediate Algebra. By J. R. BRITTON and L. C. SNIVELY. Revised Edition. Pp. 353. $3.00. 1959. (Rinehart and Co. N.Y.)

The revised edition has been augmented by the addition of a chapter on inequalities.

Hydrostatics and Mechanics. By A. E. E. MCKENZIE. Third Edition. Pp. 272. 7s. 6d. 1958. (Cambridge University Press)

In this edition accounts of scientific applications such as exploration of great depths, ascents into the stratosphere and weather forecasting have been brought up to date.

Group Theory and its applications to the Quantum Mechanics of Atomic Spectra. By E. P. WIGNER. Translated by J. J. GRIFFIN. Pp. 372. $8.80. 1959.

This translation from the famous original German Edition of 1931 contains three new chapters, discussing Racah coefficients, the time inversion operation and the classical interpretation of coefficients.

Traité des Substitutions et des Equations Algébriques. By CAMILLE JORDAN. Pp. 667. Fr. 7500. 1957. (Blanchard, Paris)

In 1870, Jordan's majestic treatise won for Galois' great ideas the fame for which they had waited forty years. This reprint is to be warmly welcomed not only for the sake of the book's historical importance, but for its continued value to the research student.

THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

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1944. La géométrie, déesse parfaite, ne dédaigne pas d'être rapprochée des tours du prestidigitateur. Elle est une rivale pour le chapeau buse dont sortent un lapin, des hochets, des rubans.—Pierre Anspach, *Aperçu de la Théorie des Polygones Réguliers*. [Per Mr. H. M. Cundy.]

1945. Un auteur est-il banal, infécond? Qu'il écrive un livre de science! Les auteurs de tels ouvrages sont comparables à une femme qui plongera dans une eau enchantée et en sortirait chargée de coquillages, de fleurs, de paillettes d'or.—Pierre Anspach, *Aperçu de la Théorie des Polygones Réguliers*. [Per Mr. H. M. Cundy.]

1946. The charge for water for a house of rateable value £50 is 10·98825% of the rateable value.—On the back of a bill from the —— —— Valley Water Co. [Per Mr. A. W. Siddons.]

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